

Periodic analogues of Dedekind sums and transformation formulas of Eisenstein series

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Abstract

In this paper, a transformation formula under modular substitutions is derived for a large class of generalized Eisenstein series. Appearing in the transformation formulae are generalizations of Dedekind sums involving the periodic Bernoulli function. Reciprocity theorems are proved for these Dedekind sums. Furthermore, as an application of the transformation formulae, relations between various infinite series and evaluations of several infinite series are deduced. Finally, we consider these sums for some special cases.

Keywords: Eisenstein series, Dedekind sums, Bernoulli numbers and polynomials.

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1 Introduction

For integers c and d with $c > 0$, the classical Dedekind sum $s(d, c)$ is defined by

$$s(d, c) = \sum_{n(\bmod c)} \left(\left(\frac{n}{c} \right) \right) \left(\left(\frac{dn}{c} \right) \right)$$

where the *sawtooth* function is defined by

$$\left((x) \right) = \begin{cases} x - [x] - 1/2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

with $[x]$ the floor function. One of the most important properties of Dedekind sums is the reciprocity formula

$$s(d, c) + s(c, d) = -\frac{1}{4} + \frac{1}{12} \left(\frac{d}{c} + \frac{c}{d} + \frac{1}{dc} \right) \quad (1)$$

whenever c and d are coprime positive integers. For several proofs of (1) and generalizations for Dedekind sums, for example, see [3, 5, 6, 7, 10, 11, 16, 17, 18, 20, 22, 23, 24, 25, 26].

Dedekind sums first arose in the transformation formulas of $\log \eta(z)$, where $\eta(z)$ denotes the Dedekind eta-function. There are several other functions such as Eisenstein series, which possess transformation formula similar to $\log \eta(z)$. Lewittes [21] has discovered a method of obtaining transformation formulas for certain generalized Eisenstein series.

In [5], Berndt gave a different account of the final part of Lewittes' proof. His new proof yielded elegant transformation formulas in which Dedekind sums or various generalizations of Dedekind sums appear. The results of [5] have been generalized in [7]. Berndt [13] derived a number of transformation formulas from the general theorem in [7]. Arising in the transformation formulae are various types of Dedekind sums, all of which satisfy reciprocity theorems.

In [6] and [10], Berndt considered a more general class of Eisenstein series and developed transformation formulae for a wide class of functions involving characters including the natural

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character generalizations of $\log \eta(z)$. In these formulas further generalizations of Dedekind sums appear. These sums involve characters and generalized Bernoulli functions, and possess reciprocity laws proved via transformation formulae.

Also in [12] and [13], Berndt has used the transformation formulas to evaluate several classes of infinite series and establish many relations between various infinite series.

The aim of this paper is to obtain a transformation formula for a very large class of Eisenstein series defined by

$$G(z, s; A_\alpha, B_\beta; r_1, r_2) = \sum_{m, n=-\infty}^{\infty}{}' \frac{f(\alpha m) f^*(\beta n)}{((m + r_1)z + n + r_2)^s}, \quad \operatorname{Re}(s) > 2, \quad \operatorname{Im}(z) > 0 \quad (2)$$

where $\{f(n)\}$ and $\{f^*(n)\}$, $-\infty < n < \infty$ are sequences of complex numbers with period $k > 0$, and $A_\alpha = \{f(\alpha n)\}$ and $B_\beta = \{f^*(\beta n)\}$. In (2), the dash $'$ means that the possible pair $m = -r_1, n = -r_2$ is excluded from the summation. Generalizations of Dedekind sums involving the periodic Bernoulli function appear in the transformation formulae. It is shown that these Dedekind sums obey reciprocity theorems. Moreover, transformation formulas contain many other interesting results as special cases. These results give the values of several interesting infinite series and yield relations between various infinite series, some of which generalize some results of [12, 13] and also found in Ramanujan's Notebooks; such as

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n(e^{n\gamma/2} + e^{-n\gamma/2})} + \sum_{n=1}^{\infty} \frac{\chi(n)}{n(e^{n\theta/2} + e^{-n\theta/2})} = \frac{\pi}{8},$$

which gives Entry 25 (vii) on p. 295 of Berndt [14] for $\theta = \gamma = \pi$ (see Example 19). Finally, we consider these Dedekind sums for some special values of A_α and B_β .

We summarize this study as follows: Section 2 is a preliminary section containing the basic definitions and notations we need. In Section 3, we show that the function $G(z, s; A_\alpha, B_\beta; r_1, r_2)$ can be analytically continued to the entire s -plane with the possible exception of a simple pole at $s = 1$. Section 4 deals with a transformation formula for the function $G(z, s; A, B; r_1, r_2)$. In Section 5, we first investigate the transformation formula for the case $s = r_1 = r_2 = 0$, in which a generalization of Dedekind sum, called periodic Dedekind sums, appears. Setting $s = 0$ and r_1, r_2 arbitrary real numbers in the transformation formula, new generalization of Dedekind sum arises, as well. Moreover, we prove the reciprocity theorems for these Dedekind sums. In Section 6, concerning with some special cases of transformation formulae allows us to present several relations between various infinite series. In Section 7, the periodic Dedekind sums is illustrated for some special values of A_α and B_β .

The definition of the Eisenstein series in (2) and the methods presented in the sequel are motivated by [6] and [10].

2 Preliminaries

In this section, we give a brief summary for the material only needed in the subsequent sections. Throughout this study we use the modular transformation $Vz = V(z) = (az + b)/(cz + d)$ where a, b, c and d are integers with $ad - bc = 1$ and $c > 0$. We use the notation $\{x\}$ for the fractional part of x , and λ_x for the characteristic function of integers. The upper half-plane $\{x + iy : y > 0\}$ will be denoted by \mathbb{H} and the upper quarter-plane $\{x + iy : x > -d/c, y > 0\}$ by \mathbb{K} . We put $e(z) = e^{2\pi iz}$ and unless otherwise stated, we use the branch of the argument defined by $-\pi \leq \arg z < \pi$.

Let $\{f(n)\} = A$, $-\infty < n < \infty$ be sequence of complex numbers with period $k > 0$. For $|t| < 2\pi/k$, the periodic Bernoulli numbers and polynomials are defined by means of the generating functions [9]

$$\sum_{n=0}^{k-1} \frac{tf(n)e^{nt}}{e^{kt} - 1} = \sum_{j=0}^{\infty} \frac{B_j(A)}{j!} t^j \quad (3)$$

and

$$\sum_{n=0}^{k-1} \frac{tf(-n)e^{(n+x)t}}{e^{kt}-1} = \sum_{j=0}^{\infty} \frac{B_j(x, A)}{j!} t^j. \quad (4)$$

Note that, when $A = I = \{1\}$, (3) and (4) reduce to ordinary Bernoulli numbers and polynomials, defined by the generating functions [4]

$$\begin{aligned} \frac{t}{e^t - 1} &= \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi, \\ \frac{te^{xt}}{e^t - 1} &= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \end{aligned} \quad (5)$$

respectively. Notice that $B_0(x) = 1$, $B_1 = -1/2$, $B_1(1) = 1/2$, $B_{2n+1} = B_{2n-1}(1/2) = 0$, $n \geq 1$, and $B_1(x) = x - 1/2$.

Throughout this paper, the n -th Bernoulli function will be denoted by $P_n(x)$ and is defined by

$$n!P_n(x) = B_n(x - [x]).$$

In particular $P_1(x) = x - [x] - 1/2$. These functions satisfy the Raabe or multiplication formula for all real x

$$P_n(x) = r^{n-1} \sum_{m=0}^{r-1} P_n\left(\frac{m+x}{r}\right) \quad (6)$$

and the reflection identity $P_n(-x) = (-1)^n P_n(x)$ except $n = 1$ and $x \in \mathbb{Z}$, in that case

$$P_1(-x) = P_1(x) = P_1(0) = -1/2.$$

The periodic Bernoulli functions $P_n(x, A)$, are functions with period k , may be defined by [9]

$$P_0(x, A) = B_0(A) = \frac{1}{k} \sum_{m=0}^{k-1} f(m) \quad (7)$$

and

$$P_n(x, A) = k^{n-1} \sum_{m=0}^{k-1} f(-m) P_n\left(\frac{m+x}{k}\right), \quad n \geq 1 \quad (8)$$

for all real x .

Define the sequence $\hat{A} = \{\hat{f}(n)\}$ by

$$\hat{f}(n) = \frac{1}{k} \sum_{j=0}^{k-1} f(j) e(-nj/k) \quad (9)$$

for $-n < \infty < n$. These are the finite Fourier series coefficients of $\{f(n)\}$. Clearly \hat{A} also has period k . Note that (9) holds if and only if

$$f(n) = \sum_{j=0}^{k-1} \hat{f}(j) e(nj/k), \quad -n < \infty < n. \quad (10)$$

3 The function $G(z, s; A_\alpha, B_\beta; r_1, r_2)$

Let A_α denote the sequence $\{f(\alpha n)\}$, $-\infty < n < \infty$, i.e., $\{f(\alpha n)\} = A_\alpha$, $\alpha \in \mathbb{Z}$. Similarly $\{f^*(\alpha n)\} = B_\alpha$, $-\infty < n < \infty$. We begin with a study of the function

$$G(z, s; A_\alpha, B_\beta; r_1, r_2) = \sum_{m, n=-\infty}^{\infty} \frac{f(\alpha m) f^*(\beta n)}{((m+r_1)z + n+r_2)^s}.$$

From the definition, we see that

$$\begin{aligned}
G(z, s; A_\alpha, B_\beta; r_1, r_2) &= \lambda_{r_1} f(-\alpha r_1) \sum_{n=-\infty}^{\infty} f^*(\beta n)(n+r_2)^{-s} + \left(\sum_{m < -r_1} \sum_{n=-\infty}^{\infty} + \sum_{m > -r_1} \sum_{n=-\infty}^{\infty} \right) \frac{f(\alpha m) f^*(\beta n)}{((m+r_1)z + n+r_2)^s} \\
&= S_1 + S_2 + S_3.
\end{aligned} \tag{11}$$

Firstly, we write S_1 as

$$\begin{aligned}
S_1 &= \lambda_{r_1} f(-\alpha r_1) \left(\sum_{n > -r_2} f^*(\beta n)(n+r_2)^{-s} + \sum_{n > r_2} f^*(-\beta n)(-n+r_2)^{-s} \right) \\
&= \lambda_{r_1} f(-\alpha r_1) (L(s, B_\beta; r_2) + e(s/2)L(s, B_{-\beta}; -r_2))
\end{aligned} \tag{12}$$

where

$$L(s; A_\beta; \theta) = \sum_{n > -\theta} f(n\beta)(n+\theta)^{-s}, \text{ for } \operatorname{Re}(s) > 1 \text{ and } \theta \text{ real.} \tag{13}$$

$L(s; A_\beta; \theta)$ can be written in terms of Hurwitz zeta function $\zeta(s, \theta)$ as follows: Setting $n = mk + j + [-\theta] + 1$, $0 \leq j \leq k-1$, $0 \leq m < \infty$ and using the fact $[\theta] + [-\theta] = \lambda_\theta - 1$, it is seen that for $\operatorname{Re}(s) > 1$

$$\begin{aligned}
L(s; A_\beta; \theta) &= \sum_{j=0}^{k-1} \sum_{m=0}^{\infty} \frac{f(\beta(j + [-\theta] + 1))}{(km + j + [-\theta] + 1 + \theta)^s} \\
&= \frac{1}{k^s} \sum_{j=1}^k f(\beta(j - [\theta] + \lambda_\theta)) \zeta\left(s, \frac{j + \{\theta\} + \lambda_\theta}{k}\right).
\end{aligned} \tag{14}$$

Since the periodic zeta-function $\zeta(s, A)$ has an analytic continuation into the entire s -plane where it is holomorphic with the possible exception of a simple pole at $s = 1$ where the residue is $\widehat{f}(0) = B_0(A)$ (see [9, Corollary 6.5]), $L(s; A_\beta; \theta)$ can be analytically continued to the entire s -plane except for $s = 1$.

Secondly, if we replace m by $-m$ and n by $-n$ in S_2 , then

$$\begin{aligned}
S_2 &= \sum_{m < -r_1} f(\alpha m) \sum_{n=-\infty}^{\infty} \frac{f^*(\beta n)}{((m+r_1)z + n+r_2)^s} \\
&= e(s/2) \sum_{m > r_1} f(-\alpha m) \sum_{n=-\infty}^{\infty} \frac{f^*(-\beta n)}{((m-r_1)z + n-r_2)^s}.
\end{aligned}$$

Using (10) and the Lipschitz summation formula given by

$$\sum_{n=1}^{\infty} (n-\tau)^{s-1} e^{2\pi i z(n-\tau)} = \frac{\Gamma(s)}{(-2\pi i)^s} \sum_{n=-\infty}^{\infty} (z+n)^{-s} e^{2\pi i n \tau}$$

where $\operatorname{Re}(s) > 1$, $z \in \mathbb{H}$ and $0 \leq \tau < 1$, we obtain

$$\begin{aligned}
\Gamma(s) \sum_{n=-\infty}^{\infty} \frac{f^*(-\beta n)}{(w+n)^s} &= \sum_{j=0}^{k-1} \frac{f^*(-\beta j)}{k^s} \Gamma(s) \sum_{n=-\infty}^{\infty} \frac{1}{(n + \frac{j+w}{k})^s} \\
&= (-2\pi i/k)^s \sum_{n=1}^{\infty} n^{s-1} e\left(\frac{nw}{k}\right) \sum_{j=0}^{k-1} f^*(-\beta j) e\left(\frac{n\beta^{-1}\beta j}{k}\right) \\
&= (-2\pi i/k)^s \sum_{n=1}^{\infty} n^{s-1} e\left(\frac{nw}{k}\right) \sum_{j=0}^{k-1} f^*(j) e\left(-n\beta^{-1}\frac{j}{k}\right) \\
&= k(-2\pi i/k)^s \sum_{n=1}^{\infty} \widehat{f}^*(\beta^{-1}n) n^{s-1} e\left(\frac{nw}{k}\right)
\end{aligned}$$

for $\text{Re}(s) > 1$, $\text{Im}(w) > 0$ and $\beta\beta^{-1} \equiv 1 \pmod{k}$. Thus, we have

$$S_2 = \frac{(-2\pi i/k)^s}{\Gamma(s)} k e(s/2) A\left(z, s; A_{-\alpha}, \widehat{B}_{\beta^{-1}}; -r_1, -r_2\right) \quad (15)$$

where

$$A(z, s; A_\alpha, A_\beta; r_1, r_2) = \sum_{m > -r_1} f(\alpha m) \sum_{n=1}^{\infty} f(\beta n) e\left(n \frac{(m+r_1)z + r_2}{k}\right) n^{s-1}.$$

Similarly we deduce that

$$S_3 = \frac{(-2\pi i/k)^s}{\Gamma(s)} k A\left(z, s; A_\alpha, \widehat{B}_{-\beta^{-1}}; r_1, r_2\right). \quad (16)$$

Combining (11), (12), (15) and (16), we conclude that

$$\begin{aligned} & G(z, s; A_\alpha, B_\beta; r_1, r_2) \\ &= \frac{(-2\pi i/k)^s k}{\Gamma(s)} \left(A\left(z, s; A_\alpha, \widehat{B}_{-\beta^{-1}}; r_1, r_2\right) + e(s/2) A\left(z, s; A_{-\alpha}, \widehat{B}_{\beta^{-1}}; -r_1, -r_2\right) \right) \\ &+ \lambda_{r_1} f(-\alpha r_1) (L(s; B_\beta; r_2) + e(s/2) L(s; B_{-\beta}; -r_2)). \end{aligned} \quad (17)$$

Since $L(s; B_\beta; r_2)$ can be analytically continued to the entire s -plane with the possible exception $s = 1$ and since $A(z, s; A_\alpha, B_\beta; r_1, r_2)$ is entire function of s , $G(z, s; A_\alpha, B_\beta; r_1, r_2)$ can be analytically continued to the entire s -plane with the possible exception $s = 1$.

For simplicity, the function $G(z, s; A_\alpha, B_\beta; 0, 0)$ will be denoted by $G(z, s; A_\alpha, B_\beta)$ and $G(z, s; A_1, B_1; r_1, r_2) = G(z, s; A, B; r_1, r_2)$.

4 Transformation Formulas

In this section, we present transformation formulas for the function $G(z, s; A_\alpha, B_\beta; r_1, r_2)$. We need the following lemma.

Lemma 1 ([21]) *Let E, F, C and D be real with E and F not both zero and $C > 0$. Then for $z \in \mathbb{H}$,*

$$\arg((Ez + F)/(Cz + D)) = \arg(Ez + F) - \arg(Cz + D) + 2\pi l,$$

where l is independent of z and $l = \begin{cases} 1, & E \leq 0 \text{ and } DE - CF > 0, \\ 0, & \text{otherwise.} \end{cases}$

Theorem 2 *Define $R_1 = ar_1 + cr_2$ and $R_2 = br_1 + dr_2$, in which r_1 and r_2 are arbitrary real numbers. Let $\rho = \rho(R_1, R_2, c, d) = \{R_2\}c - \{R_1\}d$. Suppose first that $a \equiv d \pmod{k}$. Then for $z \in \mathbb{K}$ and all s ,*

$$\begin{aligned} & (cz + d)^{-s} \Gamma(s) G(Vz, s; A, B; r_1, r_2) \\ &= \Gamma(s) G(z, s; B_{-b}, A_{-c}; R_1, R_2) - 2i\Gamma(s) \sin(\pi s) L(s; A_c; -R_2) f^*(bR_1) \lambda_{R_1} \\ &+ e(-s/2) \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(c([R_2 + d(j - \{R_1\})/c] - v)) f^*(b(\mu c + j + [R_1])) I(z, s, c, d, r_1, r_2) \end{aligned} \quad (18)$$

where $L(s; A_c; R_2)$ is given by (13) and

$$I(z, s, c, d, r_1, r_2) = \int_C u^{s-1} \frac{\exp(-((c\mu + j - \{R_1\})/ck)(cz + d)ku)}{\exp(-ku(cz + d)) - 1} \frac{\exp(((v + \{(dj + \rho)/c\})/k)ku)}{\exp(ku) - 1} du. \quad (19)$$

Here, we choose the branch of u^s with $0 < \arg u < 2\pi$. Also, C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is

the only zero of $(\exp(-ku(cz+d)) - 1)(\exp(ku) - 1)$ lying "inside" the loop, and then returning to $+\infty$ in the lower half-plane. Secondly, if $b \equiv c \equiv 0 \pmod{k}$, then for $z \in \mathbb{K}$ and all s ,

$$\begin{aligned} & (cz+d)^{-s} \Gamma(s) G(Vz, s; A, B; r_1, r_2) \\ &= \Gamma(s) G(z, s; A_d, B_a; R_1, R_2) - 2i\Gamma(s) \sin(\pi s) L(s; B_{-a}; -R_2) f(-dR_1) \lambda_{R_1} \\ &+ e(-s/2) \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-a([R_2 + d(j - \{R_1\})/c] - v + d\mu)) f(-d(j + [R_1])) I(z, s, c, d, r_1, r_2). \end{aligned} \quad (20)$$

Remark 3 Observe that the transformation formulas obtained by Berndt [5, 6, 10] are the special cases of Theorem 2.

Proof. For $z \in \mathbb{H}$ and $\operatorname{Re}(s) > 2$,

$$G(Vz, s; A, B; r_1, r_2) = \sum_{m,n=-\infty}^{\infty} f(m) f^*(n) \left\{ \frac{((M+R_1)z + N + R_2)}{cz+d} \right\}^{-s}$$

where $M = ma + nc$, $N = mb + nd$. M and N range over all pairs of integers except for the possibility $-R_1, -R_2$ as the pair m, n ranges over all pairs of integers, except for possibly the pair $-r_1, -r_2$ (since $ad - bc = 1$). Thus,

$$\begin{aligned} & G(Vz, s; A, B; r_1, r_2) \\ &= \sum_{M,N=-\infty}^{\infty} f(Md - Nc) f^*(Na - Mb) \left\{ \frac{((M+R_1)z + N + R_2)}{cz+d} \right\}^{-s} \\ &= \sum_{m,n=-\infty}^{\infty} f(-nc) f^*(-mb) \left\{ \frac{((m+R_1)z + n + R_2)}{cz+d} \right\}^{-s}, \quad a \equiv d \equiv 0 \pmod{k}, \\ &= \sum_{m,n=-\infty}^{\infty} f(dm) f^*(an) \left\{ \frac{((m+R_1)z + n + R_2)}{cz+d} \right\}^{-s}, \quad b \equiv c \equiv 0 \pmod{k}. \end{aligned}$$

Using Lemma 1, we obtain that

$$\begin{aligned} (cz+d)^{-s} G(Vz, s; A, B; r_1, r_2) &= \left(e(-s) \sum_{\substack{m+R_1 \leq 0 \\ d(m+R_1) > c(n+R_2)}} + \sum_{\substack{m,n \\ \text{otherwise}}} \right) \frac{f^*(-mb) f(-nc)}{((m+R_1)z + n + R_2)^s} \\ &= G(z, s; B_{-b}, A_{-c}; R_1, R_2) + (e(-s) - 1) g(z, s; B_{-b}, A_{-c}; R_1, R_2) \end{aligned} \quad (21)$$

where

$$g(z, s; B_{-b}, A_{-c}; R_1, R_2) = \sum_{\substack{m+R_1 \leq 0 \\ d(m+R_1) > c(n+R_2)}} \frac{f^*(-mb) f(-nc)}{((m+R_1)z + n + R_2)^s}. \quad (22)$$

In (22), we replace m by $-m$ and n by $-n$ and separate the terms with $m = R_1$. Thus

$$g(z, s; B_{-b}, A_{-c}; R_1, R_2) = e(s/2) \{ \lambda_{R_1} f^*(bR_1) L(s, A_c; -R_2) + h(z, s; B_b, A_c; -R_1, -R_2) \} \quad (23)$$

where

$$h(z, s; B_b, A_c; -R_1, -R_2) = \sum_{m > R_1} \sum_{n > R_2 + d(m-R_1)/c} \frac{f^*(mb) f(nc)}{((m-R_1)z + n - R_2)^s}.$$

Since $\operatorname{Re}((m-R_1)z + n - R_2) > 0$ for $x > -d/c$, using the Euler's integral representation of $\Gamma(s)$, we find for $z \in \mathbb{K}$ and $\operatorname{Re}(s) > 2$ that

$$\Gamma(s) h(z, s; B_b, A_c; -R_1, -R_2) = \sum_{m > R_1} \sum_{n > R_2 + d(m-R_1)/c} f(nc) f^*(mb) \int_0^{\infty} u^{s-1} \exp(-(m-R_1)zu - (n-R_2)u) du.$$

We set $m = m'c + j + [R_1] + 1$, $0 \leq m' < \infty$, $0 \leq j \leq c - 1$ and $n = n' + [R_2 + d(m - R_1)/c] + 1$. The double sum above becomes

$$\sum_{j=0}^{c-1} \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} f^*(b(m'c + j + [R_1] + 1)) f(c(n' + [R_2 + d(m'c + j - \{R_1\} + 1)/c] + 1)) \\ \times \int_0^{\infty} u^{s-1} \exp(-(m'c + j - \{R_1\} + 1)zu) \exp(-(n' + [R_2 + d(m'c + j - \{R_1\} + 1)/c] + 1 - R_2)u) du$$

Replacing $j + 1$ by j , and using that $d \equiv 0 \pmod{k}$, put $m' = mk + \mu$, $0 \leq m < \infty$, $0 \leq \mu \leq k - 1$, and put $n' = nk + v$, $0 \leq n < \infty$, $0 \leq v \leq k - 1$, we have

$$\begin{aligned} & \Gamma(s) h(z, s; B_b, A_c; -R_1, -R_2) \\ &= \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c(v + [R_2 + d(j - \{R_1\})/c] + 1)) \\ & \quad \times \int_0^{\infty} u^{s-1} \exp(-(c\mu + j - \{R_1\})zu - (v + [R_2 + d(j - \{R_1\})/c] + 1 - R_2)u) \\ & \quad \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \exp(-mku(cz + d) - nku) du \\ &= \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c(v + [R_2 + d(j - \{R_1\})/c] + 1)) \\ & \quad \times \int_0^{\infty} u^{s-1} \frac{\exp(-(c\mu + j - \{R_1\})zu)}{1 - \exp(-ku(cz + d))} \frac{\exp(-(v + 1 - R_2 + d\mu + [R_2 + d(j - \{R_1\})/c]u)}{1 - \exp(-ku)} du \\ &= - \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c([R_2 + d(j - \{R_1\})/c] - v)) \\ & \quad \times \int_0^{\infty} u^{s-1} \frac{\exp(-(c\mu + j - \{R_1\})(cz + d)ku/ck)}{\exp(-ku(cz + d)) - 1} \frac{\exp(((v + \{(dj + \rho)/c\})/k)ku)}{\exp(ku) - 1} du \\ &= \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c([R_2 + d(j - \{R_1\})/c] - v)) \frac{I(z, s, c, d, r_1, r_2)}{1 - e(s)} \end{aligned} \quad (24)$$

where

$$I(z, s, c, d, r_1, r_2) = \int_C u^{s-1} \frac{\exp(-((c\mu + j - \{R_1\})/ck)(cz + d)ku)}{\exp(-ku(cz + d)) - 1} \frac{\exp(((v + \{(dj + \rho)/c\})/k)ku)}{\exp(ku) - 1} du.$$

Here, in the next to the last step, we have multiplied the numerator and denominator by $\exp(ku)$ and then replaced $k - 1 - v$ by v . In the last step, we have used a classical method of Riemann to convert the integral over $(0, \infty)$ to a loop integral [27]. Combining (21), (23) and (24) we deduce (18). By analytic continuation the result is valid for all s . The proof of (20) is completely analogous. ■

We will also need the following theorem whose proof is similar to the proof of (18).

Theorem 4 *Under the conditions of Theorem 2, for $a \equiv d \equiv 0 \pmod{k}$ we have*

$$\begin{aligned} & (cz + d)^{-s} \Gamma(s) G(Vz, s; B_{-\beta}, A_{-\alpha}; r_1, r_2) \\ &= \Gamma(s) G(z, s; A_{\alpha b}, B_{\beta c}; R_1, R_2) - 2i\Gamma(s) \sin(\pi s) f(-\alpha b R_1) L(s, B_{-\beta c}; -R_2) \\ & \quad + e(-s/2) \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(-\alpha b(\mu c + j + [R_1])) f^*(-\beta c([R_2 + d(j - \{R_1\})/c] - v)) I(z, s, c, d, r_1, r_2), \end{aligned} \quad (25)$$

where $I(z, s, c, d, r_1, r_2)$ is given by (19).

Additionally, setting $a = d = 0$ and $c = -b = 1$ in (25), we have

$$\begin{aligned} & z^{-s} \Gamma(s) G(-1/z, s; B_{-\beta}, A_{-\alpha}; r_1, r_2) \\ &= \Gamma(s) G(z, s; A_{-\alpha}, B_{\beta}; R_1, R_2) - 2i \Gamma(s) \sin(\pi s) f(\alpha R_1) L(s, B_{-\beta}; -R_2) \\ &+ e(-s/2) \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(\alpha(\mu+1+[R_1])) f^*(-\beta([R_2]-v)) I(z, s, 1, 0, r_1, r_2). \end{aligned} \quad (26)$$

5 The periodic analogue of Dedekind sum

Our main results can be simplified when s is an integer. In this case, $I(z, s, c, d, r_1, r_2)$ can be evaluated by the residue theorem with the aid of (5). Therefore, if $s = -N$, where N is a nonnegative integer, a simple calculation yields

$$I(z, -N, c, d, r_1, r_2) = 2\pi i k^N \sum_{m+n=N+2} B_m \left(\frac{c\mu + j - \{R_1\}}{ck} \right) B_n \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) \frac{(-(cz + d))^{m-1}}{m!n!}. \quad (27)$$

From which it is seen for $s = -N$ that Theorem 2 is valid for $z \in \mathbb{H}$ by analytic continuation.

5.1 The case $s = r_1 = r_2 = 0$

For $s = r_1 = r_2 = 0$, (27) becomes

$$I(z, 0, c, d, 0, 0) = -\frac{\pi i}{cz + d} B_2 \left(\frac{v + \{dj/c\}}{k} \right) - \pi i (cz + d) B_2 \left(\frac{c\mu + j}{ck} \right) + 2\pi i B_1 \left(\frac{c\mu + j}{ck} \right) B_1 \left(\frac{v + \{dj/c\}}{k} \right).$$

To begin with, let us consider the case $a \equiv d \equiv 0 \pmod{k}$. Then, (18) can be written as

$$\begin{aligned} & \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(cz + d)^s} G(Vz, s; A, B) - G(z, s; B_{-b}, A_{-c}) \right) \\ &= -2i f^*(0) \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s, A_c; 0) + \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j)) f(c([dj/c] - v)) I(z, 0, c, d, 0, 0). \end{aligned} \quad (28)$$

We shall evaluate the triple sums in (28). Let

$$\begin{aligned} & \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j)) f(c([dj/c] - v)) \left(-\frac{\pi i}{cz + d} B_2 \left(\frac{v + \{dj/c\}}{k} \right) \right. \\ & \quad \left. - \pi i (cz + d) B_2 \left(\frac{c\mu + j}{ck} \right) + 2\pi i B_1 \left(\frac{c\mu + j}{ck} \right) B_1 \left(\frac{v + \{dj/c\}}{k} \right) \right) \\ &= T_1 + T_2 + T_3. \end{aligned}$$

Firstly,

$$\begin{aligned} T_1 &= -\pi i \frac{1}{cz + d} \sum_{j=1}^c \sum_{\mu=0}^{k-1} f^*(b(\mu c + j)) \sum_{v=0}^{k-1} f(c([dj/c] - v)) B_2 \left(\frac{v + \{dj/c\}}{k} \right) \\ &= -\pi i \frac{2}{cz + d} k B_0(B) \sum_{j=1}^c \sum_{v=0}^{k-1} f(c([dj/c] - v)) P_2 \left(\frac{v + \{dj/c\}}{k} \right) \\ &= -\pi i \frac{2}{cz + d} k B_0(B) \sum_{j=1}^c \sum_{v=0}^{k-1} f(-cv) P_2 \left(\frac{v + dj/c}{k} \right). \end{aligned}$$

For the convenience with the notation $P_r(x, A)$, denote the sum over v as $P_r(x, A_c)$, namely,

$$k^{r-1} \sum_{v=0}^{k-1} f(-cv) P_r \left(\frac{v+x}{k} \right) = P_r(x, A_c). \quad (29)$$

Note that $P_r(x, A_1) = P_r(x, A)$. Thus, for $(d, c) = 1$ and $d \equiv 0 \pmod{k}$,

$$\begin{aligned} \sum_{j=1}^c P_r(dj/c, A_c) &= k^{r-1} \sum_{v=0}^{k-1} f(-cv) \sum_{j=1}^c P_r \left(\frac{v+dj/c}{k} \right) \\ &= k^{r-1} \sum_{v=0}^{k-1} f(-cv) \sum_{j=1}^c P_r \left(\frac{v}{k} + \frac{mj}{c} \right) \\ &= k^{r-1} c^{1-r} \sum_{v=0}^{k-1} f(-cv) P_r \left(\frac{cv}{k} \right) \\ &= c^{1-r} P_r(0, A). \end{aligned}$$

Therefore, with the use of $P_r(0, A) = (-1)^r B_r(A)/r!$, for $r \geq 2$ or $r = 0$ [9, Eq. (2.12)],

$$T_1 = -\frac{\pi i}{c(cz+d)} B_0(B) B_2(A).$$

Secondly, using (7) and (6)

$$\begin{aligned} T_2 &= -\pi i(cz+d) \sum_{j=1}^c \sum_{\mu=0}^{k-1} f^*(b(\mu c + j)) B_2 \left(\frac{c\mu + j}{ck} \right) \sum_{v=0}^{k-1} f(c([dj/c] - v)) \\ &= -2\pi i(cz+d) k B_0(A) \sum_{n=0}^{ck-1} f^*(-bn) P_2 \left(\frac{n}{ck} \right) \\ &= -2\pi i(cz+d) k B_0(A) \sum_{m=0}^{k-1} \sum_{v=0}^{c-1} f^*(-bm) P_2 \left(\frac{vk+m}{ck} \right) \\ &= -\frac{2\pi i}{c} (cz+d) B_0(A) P_2(0, B_b). \end{aligned}$$

Finally,

$$\begin{aligned} &(2\pi i)^{-1} T_3 \\ &= \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j)) f(c([dj/c] - v)) B_1 \left(\frac{c\mu + j}{ck} \right) B_1 \left(\frac{v + \{dj/c\}}{k} \right) \\ &= \sum_{j=1}^{c-1} \sum_{\mu=0}^{k-1} f^*(b(\mu c + j)) P_1 \left(\frac{c\mu + j}{ck} \right) \sum_{v=0}^{k-1} f(c([dj/c] - v)) P_1 \left(\frac{v - [dj/c] + dj/c}{k} \right) \\ &\quad + \sum_{\mu=0}^{k-1} f^*(bc(\mu + 1)) B_1 \left(\frac{\mu + 1}{k} \right) \sum_{v=0}^{k-1} f(-cv) P_1 \left(\frac{v}{k} \right) \\ &= \sum_{\mu=0}^{k-1} \sum_{j=1}^c f^*(b(\mu c + j)) P_1 \left(\frac{c\mu + j}{ck} \right) \sum_{v=0}^{k-1} f(-cv) P_1 \left(\frac{v + dj/c}{k} \right) \\ &\quad + P_1(0, A_c) \sum_{\mu=1}^k f^*(bc\mu) \left\{ B_1 \left(\frac{\mu}{k} \right) - P_1 \left(\frac{\mu}{k} \right) \right\}. \end{aligned}$$

So,

$$T_3 = 2\pi i \sum_{n=1}^{ck} f^*(bn) P_1 \left(\frac{n}{ck} \right) P_1 \left(\frac{dn}{c}, A_c \right) + 2\pi i f^*(0) P_1(0, A_c).$$

Then, we find that

$$\begin{aligned} T_1 + T_2 + T_3 &= -\frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - \frac{2\pi i}{c}(cz+d)B_0(A)P_2(0, B_b) \\ &\quad + 2\pi i \sum_{n=1}^{ck} f^*(bn)P_1\left(\frac{n}{ck}\right)P_1\left(\frac{dn}{c}, A_c\right) + 2\pi i f^*(0)P_1(0, A_c). \end{aligned} \quad (30)$$

Definition 5 Let c and d be coprime integers with $d \equiv 0 \pmod{k}$ and $c > 0$. For $bc \equiv -1 \pmod{d}$, the periodic Dedekind sum $s(d, c; B_b, A_c)$ is defined by

$$s(d, c; B_b, A_c) = \sum_{n=1}^{ck} f^*(bn)P_1\left(\frac{n}{ck}\right)P_1\left(\frac{dn}{c}, A_c\right).$$

We emphasize that the sum $s(d, c; B, A)$ has been defined by Berndt [11, Section 7] without restrictions $ad - bc = 1$ and $a \equiv d \equiv 0 \pmod{k}$.

To compute $L(0; A_\beta; \theta)$ we utilize (14), the formula $\zeta(0, \theta) = 1/2 - \theta = -B_1(\theta)$, $0 < \theta \leq 1$, and $B_1(1 - \theta) = -B_1(\theta)$. We then see that

$$\begin{aligned} L(0; A_\beta; \theta) &= -\sum_{j=0}^{k-1} f(\beta(j - [\theta] + \lambda_\theta))B_1\left(\frac{j + \{\theta\} + \lambda_\theta}{k}\right) \\ &= -\sum_{j=0}^{k-2} f(\beta(j - [\theta] + \lambda_\theta))P_1\left(\frac{j + \{\theta\} + \lambda_\theta}{k}\right) - f(\beta(-1 - [\theta] + \lambda_\theta))B_1\left(1 - \frac{1 - \{\theta\} - \lambda_\theta}{k}\right) \\ &= \sum_{j=1}^{k-1} f(-\beta(j + 1 + [\theta] - \lambda_\theta))P_1\left(\frac{j + 1 - \{\theta\} - \lambda_\theta}{k}\right) + f(-\beta(1 + [\theta] - \lambda_\theta))P_1\left(\frac{1 - \{\theta\} - \lambda_\theta}{k}\right) \\ &= \sum_{j=0}^{k-1} f(-\beta j)P_1\left(\frac{j - \theta}{k}\right) = P_1(-\theta, A_\beta). \end{aligned} \quad (31)$$

Therefore, we have

$$-2if^*(0)\lim_{s \rightarrow 0} s\Gamma(s)\frac{\sin(\pi s)}{s\pi}\pi L(s; A_c; 0) = -2i\pi f^*(0)P_1(0, A_c). \quad (32)$$

Next, substituting (30) and (32) in (28) gives

$$\begin{aligned} \lim_{s \rightarrow 0} \Gamma(s) \left((cz + d)^{-s} G(Vz, s; A, B) - G(z, s; B_{-b}, A_{-c}) \right) \\ = 2\pi i s(d, c; B_b, A_c) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - \frac{2\pi i}{c}(cz+d)B_0(A)P_2(0, B_b). \end{aligned} \quad (33)$$

The second case we will consider is $b \equiv c \equiv 0 \pmod{k}$. Now (20) becomes

$$\begin{aligned} \lim_{s \rightarrow 0} \Gamma(s) \left((cz + d)^{-s} G(Vz, s; A, B) - G(z, s; A_d, B_a) \right) \\ = -2if^*(0)\lim_{s \rightarrow 0} \Gamma(s)\sin(\pi s)L(s; B_{-a}; 0) \\ + \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-a([dj/c] + v + d\mu))f(-d(\mu c + j))I(z, 0, c, d, 0, 0). \end{aligned} \quad (34)$$

Let

$$\begin{aligned} \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(-dj)f^*(-a([dj/c] - v + d\mu)) \left(-\pi i(cz+d)B_2\left(\frac{c\mu+j}{ck}\right) \right. \\ \left. - \frac{\pi i}{cz+d}B_2\left(\frac{v + \{dj/c\}}{k}\right) + 2\pi i B_1\left(\frac{c\mu+j}{ck}\right)B_1\left(\frac{v + \{dj/c\}}{k}\right) \right) \\ = T_4 + T_5 + T_6. \end{aligned}$$

Similar to T_1 , T_2 and T_3 , it can be obtained that

$$\begin{aligned} T_4 &= -\frac{2\pi i}{c}(cz+d)B_0(B)P_2(0, A_d), \\ T_5 &= -\frac{2\pi i}{c(cz+d)}B_0(B)P_2(0, A), \\ T_6 &= 2\pi i \sum_{n=1}^{ck} f(-dn)P_1\left(\frac{n}{ck}\right)P_1\left(\frac{dn}{c}, B_{-a}\right) + 2\pi i f(0)P_1(0, B_{-a}). \end{aligned}$$

Definition 6 Let c and d be coprime integers with $c \equiv 0 \pmod{k}$ and $c > 0$. For $ad \equiv 1 \pmod{c}$, the periodic Dedekind sum is defined by

$$s(d, c; A_d, B_a) = \sum_{n=1}^{ck} f(dn)P_1\left(\frac{n}{ck}\right)P_1\left(\frac{dn}{c}, B_a\right).$$

Consequently, we have

$$\begin{aligned} T_4 + T_5 + T_6 &= 2\pi i s(d, c; A_{-d}, B_{-a}) - \frac{2\pi i}{c}(cz+d)B_0(B)P_2(0, A_d) \\ &\quad - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) + 2\pi i f(0)P_1(0, B_{-a}). \end{aligned}$$

From (32), the following equality holds

$$-2if(0)\lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s)L(s; B_{-a}; 0) = -2\pi i f(0)P_1(0, B_{-a}). \quad (35)$$

Substituting these in (34) gives

$$\begin{aligned} &\lim_{s \rightarrow 0} \Gamma(s) ((cz+d)^{-s}G(Vz, s; A, B) - G(z, s; A_d, B_a)) \\ &= 2\pi i s(d, c; A_{-d}, B_{-a}) - 2\pi i \frac{cz+d}{c}B_0(B)P_2(0, A_d) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A). \end{aligned} \quad (36)$$

We summarize the results obtained above in the next theorem.

Theorem 7 Let $z \in \mathbb{H}$. If $a \equiv d \equiv 0 \pmod{k}$, then

$$\begin{aligned} &\lim_{s \rightarrow 0} \Gamma(s) ((cz+d)^{-s}G(Vz, s; A, B) - G(z, s; B_{-b}, A_{-c})) \\ &= 2\pi i s(d, c; B_b, A_c) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - 2\pi i \frac{cz+d}{c}B_0(A)P_2(0, B_b). \end{aligned} \quad (37)$$

If $b \equiv c \equiv 0 \pmod{k}$, then

$$\begin{aligned} &\lim_{s \rightarrow 0} \Gamma(s) ((cz+d)^{-s}G(Vz, s; A, B) - G(z, s; A_d, B_a)) \\ &= 2\pi i s(d, c; A_{-d}, B_{-a}) - \frac{\pi i}{c(cz+d)}B_0(B)B_2(A) - 2\pi i \frac{cz+d}{c}B_0(B)P_2(0, A_d). \end{aligned} \quad (38)$$

Now we are ready to prove a reciprocity formula for $s(d, c; B_b, A_c)$ analogous to that of the previous Dedekind sums.

Theorem 8 Let c and d be coprime positive integers with $d \equiv 0 \pmod{k}$. For $bc \equiv -1 \pmod{d}$,

$$\begin{aligned} &s(-c, d; A_c, B_{-b}) - s(d, c; B_b, A_c) \\ &= P_1(0, B_{-b})P_1(0, A_{-c}) - \frac{d}{c}B_0(A)P_2(0, B_b) - \frac{c}{d}B_0(B)P_2(0, A_c) - \frac{1}{2dc}B_0(B)B_2(A). \end{aligned}$$

Proof. Let $Tz = V(-1/z) = (bz - a)/(dz - c)$. Replacing z by $-1/z$ in (37) gives

$$\begin{aligned} & \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz - c)^s} G(V(-1/z), s; A, B) - z^{-s} G(-1/z, s; B_{-b}, A_{-c}) \right) \\ &= 2\pi i s(d, c; B_b, A_c) - \pi i z \frac{B_0(B)B_2(A)}{c(dz - c)} - 2\pi i \frac{dz - c}{cz} B_0(A)P_2(0, B_b) \end{aligned} \quad (39)$$

and replacing Vz by Tz in (38) gives

$$\begin{aligned} & \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(dz - c)^s} G(Tz, s; A, B) - G(z, s; A_{-c}, B_b) \right) \\ &= 2\pi i s(-c, d; A_c, B_{-b}) - \pi i \frac{B_0(B)B_2(A)}{d(dz - c)} - \frac{2\pi i}{d}(dz - c)B_0(B)P_2(0, A_{-c}). \end{aligned} \quad (40)$$

Lastly, the following limit needs to be computed

$$\lim_{s \rightarrow 0} \Gamma(s) (z^{-s} G(-1/z, s; B_{-b}, A_{-c}) - G(z, s; A_{-c}, B_b)).$$

For $s = r_1 = r_2 = 0$, $\beta = b$ and $\alpha = c$, (26) and (27) become

$$\begin{aligned} & \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}) - G(z, s; A_{-c}, B_b) \right) \\ &= -2if(0) \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; B_{-b}; 0) + \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(c(\mu+1)) f^*(bv) I(z, 0, 1, 0, 0, 0) \end{aligned} \quad (41)$$

and

$$I(z, 0, 1, 0, 0, 0) = -\frac{\pi i}{z} B_2\left(\frac{v}{k}\right) - \pi i z B_2\left(\frac{\mu+1}{k}\right) + 2\pi i B_1\left(\frac{\mu+1}{k}\right) B_1\left(\frac{v}{k}\right),$$

respectively. It is obvious that

$$\begin{aligned} & \sum_{\mu=0}^{k-1} f(c(\mu+1)) \sum_{v=0}^{k-1} f^*(bv) B_2\left(\frac{v}{k}\right) = 2B_0(A)P_2(0, B_b), \\ & \sum_{v=0}^{k-1} f^*(bv) \sum_{\mu=0}^{k-1} f(c(\mu+1)) B_2\left(\frac{\mu+1}{k}\right) = 2B_0(B)P_2(0, A_c) \end{aligned}$$

and

$$\sum_{v=0}^{k-1} f^*(bv) B_1\left(\frac{v}{k}\right) \sum_{\mu=0}^{k-1} f(c(\mu+1)) B_1\left(\frac{\mu+1}{k}\right) = P_1(0, B_{-b}) P_1(0, A_{-c}) + f(0) P_1(0, B_{-b}).$$

Then (41) takes the form, with the help of (35), that

$$\begin{aligned} & \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}, 0, 0) - G(z, s; A_{-c}, B_b) \right) \\ &= 2\pi i P_1(0, B_{-b}) P_1(0, A_{-c}) - \frac{2\pi i}{z} B_0(A)P_2(0, B_b) - 2\pi i z B_0(B)P_2(0, A_c). \end{aligned} \quad (42)$$

Thus, using the fact that

$$\begin{aligned} & \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz - c)^s} G(Tz, s; A, B) - G(z, s; A_{-c}, B_b) \right) \\ &= \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz - c)^s} G(V(-1/z), s; A, B) - \frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}) \right) \\ &+ \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}) - G(z, s; A_{-c}, B_b) \right), \end{aligned}$$

it follows from (39), (40) and (42) that

$$\begin{aligned} & s(-c, d; A_c, B_{-b}) - s(d, c; B_b, A_c) \\ &= P_1(0, B_{-b}) P_1(0, A_{-c}) - \frac{1}{2dc} B_0(B) B_2(A) - \frac{1}{z} B_0(A) P_2(0, B_b) \\ & \quad - z B_0(B) P_2(0, A_c) - \frac{1}{cz} (dz - c) B_0(A) P_2(0, B_b) + \frac{1}{d} (dz - c) B_0(B) P_2(0, A_{-c}). \end{aligned}$$

From this, the reciprocity formula follows by setting $z = c/d$. ■

Remark 9 The reciprocity formula may be stated as

$$\begin{aligned} & s(c, d; A_c, B_b) + s(d, c; B_b, A_c) \\ &= -P_1(0, B_{-b}) P_1(0, A_{-c}) + \frac{1}{2dc} B_0(B) B_2(A) + \frac{d}{c} B_0(A) P_2(0, B_b) + \frac{c}{d} B_0(B) P_2(0, A_c) - f(0) P_1(0, B). \end{aligned}$$

Indeed, if we use that for $x \notin \mathbb{Z}$,

$$P_1(-x, B_{-b}) = \sum_{v=0}^{k-1} f^*(-bv) P_1\left(-\frac{v+x}{k}\right) = -P_1(x, B_b)$$

and for $x \in \mathbb{Z}$,

$$\begin{aligned} P_1(-x, B_{-b}) &= \sum_{\substack{v=0 \\ v \not\equiv x \pmod{k}}}^{k-1} f^*(bv) P_1\left(\frac{v-x}{k}\right) + f^*(xb) P_1(0) \\ &= - \sum_{\substack{v=0 \\ v \not\equiv -x \pmod{k}}}^{k-1} f^*(-bv) P_1\left(\frac{v+x}{k}\right) + f^*(xb) P_1(0) \\ &= - \sum_{v=0}^{k-1} f^*(-bv) P_1\left(\frac{v+x}{k}\right) + 2f^*(xb) P_1(0) = -P_1(x, B_b) - f^*(xb), \end{aligned}$$

then we deduce that

$$s(-c, d; A_c, B_{-b}) = -s(c, d; A_c, B_b) - f(0) P_1(0, B), \quad (43)$$

where we have used that $d \equiv 0 \pmod{k}$ and $bc \equiv -1 \pmod{k}$.

Remark 10 Berndt [11, Theorem 7.3] has derived a reciprocity formula for the sum $s(c, d; A, B)$ without restriction $d \equiv 0 \pmod{k}$ by using Riemann–Stieltjes integral.

5.2 The case $s = 0$, and r_1 and r_2 arbitrary

Let $s = 0$, and r_1 and r_2 arbitrary. From (27) we have

$$\begin{aligned} I(z, 0, c, d, R_1, R_2) &= -\frac{\pi i}{cz + d} B_2\left(\frac{v + \{(dj + \rho)/c\}}{k}\right) - \pi i(cz + d) B_2\left(\frac{c\mu + j - \{R_1\}}{ck}\right) \\ & \quad + 2\pi i B_1\left(\frac{c\mu + j - \{R_1\}}{ck}\right) B_1\left(\frac{v + \{(dj + \rho)/c\}}{k}\right). \end{aligned} \quad (44)$$

Let us consider the case $a \equiv d \equiv 0 \pmod{k}$. Then, (18) can be written as

$$\begin{aligned} & \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(cz + d)^s} G(Vz, s; A, B; r_1, r_2) - G(z, s; B_{-b}, A_{-c}; R_1, R_2) \right) \\ &= -2i f^*(bR_1) \lambda_{R_1} \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; A_c; -R_2) \\ & \quad + \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c([R_2 + d(j - \{R_1\})/c] - v)) I(z, 0, c, d, R_1, R_2). \end{aligned} \quad (45)$$

Now we evaluate the triple sums in (45). Let

$$\begin{aligned}
& \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c([R_2 + d(j - \{R_1\})/c] - v)) \\
& \times \left(-\frac{\pi i}{cz + d} B_2 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) - \pi i(cz + d) B_2 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) \right. \\
& \quad \left. + 2\pi i B_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) B_1 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) \right) \\
& = T_7 + T_8 + T_9.
\end{aligned}$$

Since $(dj + \rho)/c = d(j - \{R_1\})/c + R_2 - [R_2]$, we conclude that

$$\begin{aligned}
T_7 &= -\frac{\pi i}{cz + d} \sum_{j=1}^c \sum_{v=0}^{k-1} f(c([R_2 + d(j - \{R_1\})/c] - v)) B_2 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) \sum_{\mu=0}^{k-1} f^*(b(\mu c + j + [R_1])) \\
&= -\frac{2\pi i}{cz + d} k B_0(B) \sum_{j=1}^c \sum_{v=0}^{k-1} f(-cv) P_2 \left(\frac{v + R_2}{k} + \frac{d(j - \{R_1\})}{ck} \right).
\end{aligned}$$

Thus, for $(d, c) = 1$ and $d = mk$ (since $d \equiv 0 \pmod{k}$) it follows from (6) that

$$\begin{aligned}
T_7 &= -\frac{2\pi i}{c(cz + d)} k B_0(B) \sum_{v=0}^{k-1} f(-cv) P_2 \left(\frac{cv + cR_2 - mkR_1}{k} + m[R_1] \right) \\
&= -\frac{2\pi i}{c(cz + d)} B_0(B) P_2(cR_2 - dR_1, A).
\end{aligned}$$

Again utilizing (6) we have

$$\begin{aligned}
T_8 &= -\pi i(cz + d) \sum_{j=1}^c \sum_{\mu=0}^{k-1} f^*(b(\mu c + j + [R_1])) B_2 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) \\
& \quad \times \sum_{v=0}^{k-1} f((c[R_2 + d(j - \{R_1\})/c] - v)) \\
&= -2\pi i(cz + d) k B_0(A) \sum_{n=1}^{ck} f^*(bn) P_2 \left(\frac{n - R_1}{ck} \right) \\
&= -2\pi i(cz + d) k B_0(A) \sum_{v=0}^{c-1} \sum_{r=1}^k f^*(br) P_2 \left(\frac{v}{c} + \frac{r - R_1}{ck} \right) \\
&= -\frac{2\pi i}{c} (cz + d) B_0(A) P_2(R_1, B_b).
\end{aligned}$$

We now consider

$$\begin{aligned}
(2\pi i)^{-1} T_9 &= \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c[R_2 + d(j - \{R_1\})/c] - v) \\
& \quad \times B_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) B_1 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right).
\end{aligned}$$

We want to replace $B_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right)$ by $P_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right)$, this is valid except for $R_1 \in \mathbb{Z}$, $\mu = k-1$ and $j = c$. If $R_1 \in \mathbb{Z}$, firstly separate the term $j = c$, then write $P_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right)$ in place of

$B_1 \left(\frac{c\mu+j-\{R_1\}}{ck} \right)$, later add and subtract the term $j = c$. Then

$$T_9 = 2\pi i \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(b(\mu c + j + [R_1])) f(c([R_2 + dj/c] - v)) P_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) P_1 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) \\ + 2\pi i P_1(R_2, A_c) f^*(bR_1).$$

Therefore, for arbitrary real number R_1 , we have

$$T_9 = 2\pi i \sum_{j=1}^c \sum_{\mu=0}^{k-1} f^*(b(\mu c + j + [R_1])) P_1 \left(\frac{c\mu + j - \{R_1\}}{ck} \right) \\ \times \sum_{v=0}^{k-1} f(c([R_2 + d(j - \{R_1\})/c] - v)) P_1 \left(\frac{v + \{(dj + \rho)/c\}}{k} \right) + 2\pi i P_1(R_2, A_c) f^*(bR_1) \lambda_{R_1} \\ = 2\pi i \sum_{n=1}^{ck} f^*(bn) P_1 \left(\frac{n - R_1}{ck} \right) P_1 \left(\frac{d(n - R_1)}{c} + R_2, A_c \right) + 2\pi i P_1(R_2, A_c) f^*(bR_1) \lambda_{R_1}.$$

Definition 11 Let c and d be coprime integers with $d \equiv 0 \pmod{k}$ and $c > 0$. For $bc \equiv -1 \pmod{d}$, the generalized periodic Dedekind sum $s(d, c; A_b, A_c; x, y)$ is defined by

$$s(d, c; A_b, A_c; x, y) = \sum_{n=1}^{ck} f(bn) P_1 \left(\frac{n + y}{ck} \right) P_1 \left(\frac{d(n + y)}{c} + x, A_c \right).$$

Note that $s(d, c; A_b, A_c; 0, 0) = s(d, c; A_b, A_c)$. Moreover $s(d, c; A_b, A_c; x, y)$ is the natural generalization of the generalized Dedekind sum $s(d, c; x, y)$ [24, 5] and of the generalized Dedekind character sum $s(d, c; \chi, \bar{\chi}; x, y)$ [6].

So, we have obtained that

$$T_7 + T_8 + T_9 = 2\pi i s(d, c; B_b, A_c, R_2, -R_1) - \frac{2\pi i}{c(cz + d)} B_0(B) P_2(cR_2 - dR_1, A) \\ - \frac{2\pi i}{c} (cz + d) B_0(A) P_2(R_1, B_b) + 2\pi i P_1(R_2, A_c) f^*(bR_1) \lambda_{R_1}. \quad (46)$$

On the other hand, (31) gives

$$\lim_{s \rightarrow 0} L(s; A_c; -R_2) = P_1(R_2, A_c)$$

and then

$$-2i\lambda_{R_1} f^*(bR_1) \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; A_c; -R_2) = -2i\pi \lambda_{R_1} f^*(bR_1) P_1(R_2, A_c). \quad (47)$$

Next, combining (45), (46) and (47) yields

$$\lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(cz + d)^s} G(Vz, s; A, B; r_1, r_2) - G(z, s; B_{-b}, A_{-c}; R_1, R_2) \right) \\ = 2\pi i s(d, c; B_b, A_c, R_2, -R_1) - \frac{2\pi i}{c(cz + d)} B_0(B) P_2(cR_2 - dR_1, A) - \frac{2\pi i}{c} (cz + d) B_0(A) P_2(R_1, B_b). \quad (48)$$

Now, let us consider the case $b \equiv c \equiv 0 \pmod{k}$. Then, from (44), (20) can be written as

$$\lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(cz + d)^s} G(Vz, s; A, B; r_1, r_2) - G(z, s; A_d, B_a; R_1, R_2) \right) \\ = -2if(-dR_1) \lambda_{R_1} \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; B_{-a}; -R_2) \\ + \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-a([R_2 + d(j - \{R_1\})/c] - v + d\mu)) f(-d(j + [R_1])) I(z, 0, c, d, R_1, R_2). \quad (49)$$

Similar to (47), we have

$$2i\lambda_{R_1}f(-dR_1)\lim_{s\rightarrow 0}\Gamma(s)\sin(\pi s)L(s;B_{-a};-R_2)=-2\pi i\lambda_{R_1}f(-dR_1)P_1(R_2,B_{-a}). \quad (50)$$

Let

$$\begin{aligned} & \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-a([R_2+d(j-\{R_1\})/c]-v+d\mu))f(-d(j+[R_1])) \\ & \times \left(-\frac{\pi i}{cz+d} B_2\left(\frac{v+\{(dj+\rho)/c\}}{k}\right) - \pi i(cz+d)B_2\left(\frac{c\mu+j-\{R_1\}}{ck}\right) \right. \\ & \quad \left. + 2\pi i B_1\left(\frac{c\mu+j-\{R_1\}}{ck}\right) B_1\left(\frac{v+\{(dj+\rho)/c\}}{k}\right) \right) \\ & = T_{10} + T_{11} + T_{12}. \end{aligned}$$

To compute T_{10} , we first evaluate the sums over μ and over v , respectively, and then use the facts $B_2(\{x\}) = 2P_2(x)$ and $\rho = c\{R_2\} - d\{R_1\}$, to find that

$$\begin{aligned} T_{10} &= -\frac{\pi i}{cz+d} \sum_{j=1}^c \sum_{v=0}^{k-1} f(-d(j+[R_1]))B_2\left(\frac{v+\{(dj+\rho)/c\}}{k}\right) \\ & \times \sum_{\mu=0}^{k-1} f^*(-a([R_2+d(j-\{R_1\})/c]-v+d\mu)) \\ &= -\frac{2\pi i}{cz+d} B_0(B) \sum_{j=1}^c f(-dj)P_2\left(\frac{d(j-R_1)}{c} + \{R_2\}\right). \end{aligned}$$

Now, setting $c = qk$ (since $c \equiv 0 \pmod{k}$), then taking $j \rightarrow \mu k + r$, $0 \leq \mu \leq q-1$, $1 \leq r \leq k$, gives

$$\begin{aligned} T_{10} &= -\frac{2\pi i}{cz+d} B_0(B) \sum_{\mu=0}^{q-1} \sum_{r=1}^k f(-dr)P_2\left(\frac{d\mu}{q} + \frac{d(r-R_1)}{qk} + R_2\right) \\ &= -\frac{2\pi i}{cz+d} \frac{B_0(B)}{q} \sum_{r=1}^k f(-dr)P_2\left(\frac{dr-dR_1+cR_2}{k}\right) \\ &= -\frac{2\pi i}{c(cz+d)} B_0(B) P_2(cR_2-dR_1, A). \end{aligned}$$

Secondly, for computing T_{11} , evaluate the sums over v and over μ , respectively, to find

$$\begin{aligned} T_{11} &= -\pi i(cz+d) \sum_{j=1}^c \sum_{\mu=0}^{k-1} f(-d(j+[R_1]))B_2\left(\frac{c\mu+j-\{R_1\}}{ck}\right) \\ & \times \sum_{v=0}^{k-1} f^*(-a([R_2+d(j-\{R_1\})/c]-v+d\mu)) \\ &= -\frac{2\pi i}{c}(cz+d)B_0(B)P_2(-R_1, A_d). \end{aligned}$$

After similar arguments in evaluating T_9 , we deduce for arbitrary real number R_1 that

$$\begin{aligned}
T_{12} &= 2\pi i \sum_{j=1}^c \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-a([R_2 + d(j - \{R_1\})/c] - v + d\mu)) f(-d(j + [R_1])) \\
&\quad \times P_1\left(\frac{c\mu + j - \{R_1\}}{ck}\right) P_1\left(\frac{v + \{(dj + \rho)/c\}}{k}\right) + 2\pi i P_1(R_2, B_{-a}) f(-dR_1) \lambda_{R_1} \\
&= 2\pi i \sum_{n=1}^{ck} f(-d(n + [R_1])) P_1\left(\frac{n - \{R_1\}}{ck}\right) \sum_{v=0}^{k-1} f^*(av) P_1\left(\frac{v + d\frac{n - \{R_1\}}{c} + R_2}{k}\right) \\
&\quad + 2\pi i P_1(R_2, B_{-a}) f(-dR_1) \lambda_{R_1} \\
&= 2\pi i \sum_{n=1}^{ck} f(-dn) P_1\left(\frac{n - R_1}{ck}\right) P_1\left(\frac{d(n - R_1)}{c} + R_2, B_{-a}\right) + 2\pi i P_1(R_2, B_{-a}) f(-dR_1) \lambda_{R_1}.
\end{aligned}$$

Definition 12 Let c and d be coprime integers with $c \equiv 0 \pmod{k}$ and $c > 0$. For $ad \equiv 1 \pmod{c}$, the generalized periodic Dedekind sum $s(d, c; A_d, B_a; x, y)$ is defined by

$$s(d, c; A_d, B_a; x, y) = \sum_{n=1}^{ck} f(dn) P_1\left(\frac{n + y}{ck}\right) P_1\left(\frac{d(n + y)}{c} + x, B_a\right).$$

So, we have obtained

$$\begin{aligned}
T_{10} + T_{11} + T_{12} &= 2\pi i s(d, c; A_{-d}, B_{-a}; R_2, -R_1) - \frac{2\pi i}{c(cz + d)} B_0(B) P_2(cR_2 - dR_1, A) \\
&\quad - \frac{2\pi i}{c}(cz + d) B_0(B) P_2(-R_1, A_d) + 2\pi i P_1(R_2, B_{-a}) f(-dR_1) \lambda_{R_1}. \quad (51)
\end{aligned}$$

Combining (49), (50) and (51) yields

$$\begin{aligned}
&\lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(cz + d)^s} G(Vz, s; A, B; r_1, r_2) - G(z, s; A_d, B_a; R_1, R_2) \right) \\
&= 2\pi i s(d, c; A_{-d}, B_{-a}; R_2, -R_1) - \frac{2\pi i}{c(cz + d)} B_0(B) P_2(cR_2 - dR_1, A) \\
&\quad - \frac{2\pi i}{c}(cz + d) B_0(B) P_2(-R_1, A_d). \quad (52)
\end{aligned}$$

We now derive a reciprocity formula for $s(d, c; B_b, A_c; R_2, R_1)$.

Theorem 13 Let c and d be coprime positive integers with $d \equiv 0 \pmod{k}$. For $bc \equiv -1 \pmod{d}$,

$$\begin{aligned}
&s(-c, d; A_c, B_{-b}; -R_1, -R_2) - s(d, c; B_b, A_c; R_2, -R_1) \\
&= P_1(R_2, B_{-b}) P_1(-R_1, A_{-c}) - \frac{1}{cd} B_0(B) P_2(cR_2 - dR_1, A) - \frac{d}{c} B_0(A) P_2(R_2, B_{-b}) - \frac{c}{d} B_0(B) P_2(R_1, A_c).
\end{aligned}$$

Proof. Replacing z by $-1/z$ in (48),

$$\begin{aligned}
&\lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz - c)^s} G(Tz, s; A, B; r_1, r_2) - z^{-s} G(-1/z, s; B_{-b}, A_{-c}; R_1, R_2) \right) \\
&= 2\pi i s(d, c; B_b, A_c; R_2, -R_1) - \frac{2\pi i}{c} \frac{z}{(dz - c)} B_0(B) P_2(cR_2 - dR_1, A) - \frac{2\pi i}{cz} (dz - c) B_0(A) P_2(R_1, B_b). \quad (53)
\end{aligned}$$

Replacing Vz by Tz in (52),

$$\begin{aligned}
&\lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{(dz - c)^s} G(Tz, s; A, B; r_1, r_2) - G(z, s; A_{-c}, B_b; R_2, -R_1) \right) \\
&= 2\pi i s(-c, d; A_c, B_{-b}; -R_1, -R_2) - \frac{2\pi i}{d} \frac{1}{dz - c} B_0(B) P_2(cR_2 - dR_1, A) - \frac{2\pi i}{d} (dz - c) B_0(B) P_2(-R_2, A_{-c}). \quad (54)
\end{aligned}$$

Computing the equation (26) for $\beta = b$, $\alpha = c$ and the limiting case $s = 0$ gives

$$\begin{aligned}
& \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}, R_1, R_2) - G(z, s; A_{-c}, B_b; R_2, -R_1) \right) \\
&= -2if(cR_1)\lambda_{R_1} \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; B_{-b}; -R_2) \\
&+ \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f(c(\mu+1+[R_1])) f^*(-b([R_2]-v)) I(z, 0, 1, 0, R_1, R_2)
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
& I(z, 0, 1, 0, R_1, R_2) \\
&= -\frac{\pi i}{z} B_2 \left(\frac{v + \{R_2\}}{k} \right) - \pi i z B_2 \left(\frac{\mu + 1 - \{R_1\}}{k} \right) + 2\pi i B_1 \left(\frac{\mu + 1 - \{R_1\}}{k} \right) B_1 \left(\frac{v + \{R_2\}}{k} \right).
\end{aligned}$$

As in before, we have

$$\begin{aligned}
& \sum_{v=0}^{k-1} \sum_{\mu=0}^{k-1} f(c(\mu+1+[R_1])) f^*(-b([R_2]-v)) B_2 \left(\frac{v + \{R_2\}}{k} \right) = 2B_0(A) P_2(R_2, B_{-b}), \\
& \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} f^*(-b([R_2]-v)) f(c(\mu+1+[R_1])) B_2 \left(\frac{\mu + 1 - \{R_1\}}{k} \right) = 2B_0(B) P_2(R_1, A_c)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{v=0}^{k-1} f^*(-b([R_2]-v)) B_1 \left(\frac{v + \{R_2\}}{k} \right) \sum_{\mu=0}^{k-1} f(c(\mu+1+[R_1])) B_1 \left(\frac{\mu + 1 - \{R_1\}}{k} \right) \\
&= P_1(R_2, B_{-b}) P_1(-R_1, A_{-c}) + f(cR_1)\lambda_{R_1}.
\end{aligned}$$

From (47) we have

$$-2if(cR_1)\lambda_{R_1} \lim_{s \rightarrow 0} \Gamma(s) \sin(\pi s) L(s; B_{-b}; -R_2) = -2\pi i f(cR_1)\lambda_{R_1} P_1(R_2, B_{-b}).$$

Thus, combining above in (55) gives

$$\begin{aligned}
& \lim_{s \rightarrow 0} \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}, R_1, R_2) - G(z, s; A_{-c}, B_b; R_2, -R_1) \right) \\
&= 2\pi i P_1(R_2, B_{-b}) P_1(-R_1, A_{-c}) - \frac{2\pi i}{z} B_0(A) P_2(R_2, B_{-b}) - 2\pi i z B_0(B) P_2(R_1, A_c).
\end{aligned} \tag{56}$$

Taking into account that

$$\begin{aligned}
& \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz-c)^s} G(Tz, s; A, B; r_1, r_2) - G(z, s; A_{-c}, B_b; R_2, -R_1) \right) \\
&= \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{(dz-c)^s} G(V(-1/z), s; A, B; r_1, r_2) - \frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}; R_1, R_2) \right) \\
&+ \lim_{s \rightarrow 0} z^s \Gamma(s) \left(\frac{1}{z^s} G(-1/z, s; B_{-b}, A_{-c}; R_1, R_2) - G(z, s; A_{-c}, B_b; R_2, -R_1) \right),
\end{aligned}$$

(53), (54) and (56) entail the reciprocity formula by setting $z = c/d$. ■

6 Some series relations

In this section we illustrate some transformation formulas of $A(z, s; A_\alpha, B_\beta) := A(z, s; A_\alpha, B_\beta; 0, 0)$ for the special values of $A = \{f(n)\}$ and $B = \{f^*(n)\}$ by using (17) and (18) or (37). As

applications of these formulas we present several relations between various infinite series. For the rest of this paper we assume that $a \equiv d \equiv 0 \pmod{k}$ with $ad - bc = 1$ and χ_0 will denote the principle character of modulus $k \geq 2$, i.e.,

$$\chi_0(n) = \begin{cases} 1, & \text{if } (n, k) = 1 \\ 0, & \text{if } (n, k) > 1. \end{cases}$$

We first write $\Gamma(s)G(z, s; A, B)$ and $\Gamma(s)G(z, s; B_{-b}, A_{-c})$ in terms of $A(z, s; A_\alpha, B_\beta)$. Put $L(s; A_c) = L(s; A_\alpha; 0)$. From (17), for $\alpha = \beta = 1$ and $r_1 = r_2 = 0$,

$$\begin{aligned} \Gamma(s)G(z, s; A, B) &= (-2\pi i/k)^s k \left(A(z, s; A, \widehat{B}_{-1}) + e(s/2)A(z, s; A_{-1}, \widehat{B}) \right) \\ &\quad + \Gamma(s)f(0)(L(s; B) + e(s/2)L(s; B_{-1})) \end{aligned} \quad (57)$$

and for $\alpha = -c, \beta = -b$ and $r_1 = r_2 = 0$,

$$\begin{aligned} \Gamma(s)G(z, s; B_{-b}, A_{-c}) &= (-2\pi i/k)^s k \left(A(z, s; B_{-b}, \widehat{A}_{c-1}) + e(s/2)A(z, s; B_b, \widehat{A}_{-c-1}) \right) \\ &\quad + \Gamma(s)f^*(0)(L(s; A_{-c}) + e(s/2)L(s; A_c)). \end{aligned} \quad (58)$$

1. Let $A = \chi_1 = \{\chi_1(n)\}$ and $B = \chi_2 = \{\chi_2(n)\}$, where χ_1 and χ_2 are Dirichlet characters of modulus k . It is clear from (9) that

$$\widehat{f}(n) = \frac{1}{k} \sum_{v=0}^{k-1} \chi(v) e^{-2\pi i n v/k} = \frac{1}{k} G(-n, \chi), \quad (59)$$

where $G(n, \chi)$ is the Gauss sum. Then

$$\widehat{A} = \{\chi_1(-1) G(n, \chi_1)/k\} = \chi_1(-1) G_1/k \text{ and } \widehat{B} = \{\chi_2(-1) G(n, \chi_2)/k\} = \chi_2(-1) G_2/k.$$

Thus, we deduce from (37) after using (57) and (58) and simplifying that

$$\begin{aligned} &\left(A(Vz, 0; \chi_1, G_2) - \chi_1(c) \chi_2(b) A(z, 0; \chi_2, G_1) \right) (1 + \chi_1(-1) \chi_2(-1)) \\ &= 2\pi i \chi_1(c) \chi_2(b) s(d, c; \chi_2, \chi_1) - \frac{\pi i}{c(cz + d)} B_0(\chi_2) B_2(\chi_1) - 2\pi i \frac{\chi_2(b)}{c} (cz + d) B_0(\chi_1) P_2(0, \chi_2). \end{aligned} \quad (60)$$

Notice that $P_m(x, \chi)$ and $B_m(\chi)$ correspond to the generalized Bernoulli functions and numbers denoted by $\chi(-1) B_m(x, \overline{\chi})$ and $\chi(-1) B_m(\overline{\chi})$, respectively, given in [8]. In the sequel we need the relations

$$B_{2m+1}(\chi) = 0 \text{ for even } \chi, B_{2m}(\chi) = 0 \text{ for odd } \chi, \text{ and } B_0(\chi) = 0 \text{ for } \chi \neq \chi_0. \quad (61)$$

If χ_1 and χ_2 are non-principle Dirichlet characters with $\chi_1(-1) \chi_2(-1) = 1$, then, by (61), (60) reduces to

$$A(Vz, 0; \chi_1, G_2) - \chi_1(c) \chi_2(b) A(z, 0; \chi_2, G_1) = \pi i \chi_1(c) \chi_2(b) s(d, c; \chi_2, \chi_1).$$

For a principle character χ_0 the Gauss sum reduces to the Ramanujan sum

$$c_k(n) := \sum_{\substack{v=1 \\ (v, k)=1}}^k e^{2\pi i n v/k}.$$

Supposing $\chi_1 = \chi_2 = \chi_0$, (60) takes the form

$$A(Vz, 0; \chi_0, c_k) - A(z, 0; \chi_0, c_k) = \pi i s(d, c; \chi_0, \chi_0) - \frac{\pi i}{2ck} \left(\frac{1}{cz + d} + cz + d \right) \phi(k) B_2(\chi_0),$$

where we have used that $B_0(\chi_0) = \phi(k)/k$ by (7). Here $\phi(k)$ stands for the Euler *phi* function, i.e., the number of positive integers not exceeding k which are relatively prime to k .

Now we utilize (18) instead of (37), with the use of (27) for $s = -2N$, where N is an integer. Using (57) and (58) in (18), with $Vz = -1/z$, and assuming that $\chi_1(-1)\chi_2(-1) = 1$, we find that

$$\begin{aligned} & z^{2N} A(-1/z, -2N; \chi_1, G_2) - \chi_1(-1) A(z, -2N; \chi_2, G_1) \\ &= \frac{(2\pi i)^{2N+1}}{2} \sum_{m=0}^{2N+2} \sum_{\mu=0}^{k-1} \sum_{v=0}^{k-1} \frac{\chi_1(-v)\chi_2(-\mu-1)}{m!(2N+2-m)!} B_{2N+2-m}\left(\frac{v}{k}\right) B_m\left(\frac{\mu+1}{k}\right) (-z)^{m-1} \\ &= -\frac{(2\pi i)^{2N+1}}{2k^{2N}} \sum_{m=0}^{2N+2} (-1)^m \frac{B_m(\chi_2)}{m!} \frac{B_{2N+2-m}(\chi_1)}{(2N+2-m)!} z^{m-1}, \end{aligned} \quad (62)$$

which can be equally written as

$$\begin{aligned} & z^{2N} \sum_{n=1}^{\infty} \frac{G(n, \chi_2) G(-n/z, \chi_1)}{1 - e^{-2\pi i n/z}} n^{-2N-1} - \chi_1(-1) \sum_{n=1}^{\infty} \frac{G(n, \chi_1) G(nz, \chi_2)}{1 - e^{2\pi i n z}} n^{-2N-1} \\ &= -\frac{(2\pi i)^{2N+1}}{2k^{2N}} \sum_{m=0}^{2N+2} (-1)^m \frac{B_m(\chi_2)}{m!} \frac{B_{2N+2-m}(\chi_1)}{(2N+2-m)!} z^{m-1}. \end{aligned} \quad (63)$$

Indeed, setting $m = v + hk$, $0 \leq v \leq k-1$, $0 \leq h < \infty$, in the double sum

$$A(z, -2N; \chi_1, G_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_1(m) G(n, \chi_2) e^{2\pi i n m z/k} n^{-2N-1}$$

we have

$$\begin{aligned} A(z, -2N; \chi_1, G_2) &= \sum_{n=1}^{\infty} G(n, \chi_2) n^{-2N-1} \sum_{v=0}^{k-1} \chi_1(v) e^{2\pi i v n z/k} \sum_{h=0}^{\infty} e^{2\pi i n z h} \\ &= \sum_{n=1}^{\infty} \frac{G(n, \chi_2) G(nz, \chi_1)}{1 - e^{2\pi i n z}} n^{-2N-1}. \end{aligned} \quad (64)$$

Let $z = \pi i/k\gamma$ in (63), where $\gamma > 0$, and determine $\theta > 0$ by $\gamma\theta = \pi^2/k^2$. Then we see that

$$\begin{aligned} & \gamma^{-N} \sum_{n=1}^{\infty} \frac{G(n, \chi_2) G(ink\gamma/\pi, \chi_1)}{1 - e^{-2nk\gamma}} n^{-2N-1} \\ & - (-\theta)^{-N} \chi_1(-1) \sum_{n=1}^{\infty} \frac{G(n, \chi_1) G(ink\theta/\pi, \chi_2)}{1 - e^{-2nk\theta}} n^{-2N-1} \\ &= -k^{2N} \sum_{m=0}^{2N+2} (-i)^m \frac{B_m(\chi_2)}{m!} \frac{B_{2N+2-m}(\chi_1)}{(2N+2-m)!} \gamma^{N+1-m/2} \theta^{m/2}. \end{aligned} \quad (65)$$

Setting $\chi_1 = \chi_2 = \chi$, $\gamma = \theta = \pi/k$ in (65) or $z = i$ in (63) and assuming that $\chi(-1)(-1)^N = -1$ give

$$\sum_{n=1}^{\infty} \frac{G(n, \chi) G(in, \chi)}{1 - e^{-2\pi n}} n^{-2N-1} = -\frac{k(2\pi/k)^{2N+1}}{4} \sum_{m=0}^{2N+2} (-i)^m \frac{B_m(\chi)}{m!} \frac{B_{2N+2-m}(\chi)}{(2N+2-m)!}. \quad (66)$$

As a consequence of (66) we may write

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{G(n, \chi) G(in, \chi)}{1 - e^{-2\pi n}} n^{-1} = \frac{\pi i}{2} B_1(\chi) B_1(\chi), \text{ for odd } \chi, \\ & \sum_{n=1}^{\infty} \frac{G(n, \chi) G(in, \chi)}{1 - e^{-2\pi n}} n = -\frac{k^2}{8\pi} B_0(\chi) B_0(\chi), \text{ for even } \chi, \\ & \sum_{n=1}^{\infty} \frac{G(n, \chi) G(in, \chi)}{1 - e^{-2\pi n}} n^{2M-1} = 0, \text{ for } \chi(-1)(-1)^M = -1, M \geq 2. \end{aligned}$$

Now differentiate both sides of (62) with respect to z and then put $\chi_1 = \chi_2 = \chi$ and $z = i$. Under the assumption $\chi(-1)(-1)^N = 1$, we obtain

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi(m) G(n, \chi) e^{-2\pi nm/k} n^{-2N-1} \left(N + \frac{2\pi mn}{k} \right) \\ &= -\frac{k(2\pi/k)^{2N+1}}{4} \sum_{m=0}^{2N+2} (-i)^m (m-1) \frac{B_m(\chi_2)}{m!} \frac{B_{2N+2-m}(\chi_1)}{(2N+2-m)!}. \end{aligned} \quad (67)$$

Now we examine (65) for $\chi_1 = \chi_2 = \chi$.

Example 14 Let, in (65), $\chi_1 = \chi_2 = \chi$ be the primitive character of modulus $k = 4$ defined by

$$\chi(n) = \begin{cases} 0, & n \text{ even}, \\ 1, & n \equiv 1 \pmod{4}, \\ -1, & n \equiv 3 \pmod{4}. \end{cases} \quad (68)$$

Since $B_m(1-x) = (-1)^m B_m(x)$ and

$$B_m\left(\frac{1}{4}\right) = 2^{-m} B_m\left(\frac{1}{2}\right) - m4^{-m} E_{m-1}, m \geq 1, \quad (69)$$

where E_m is the m -th Euler number [1, p. 806], we have, for odd m , that

$$B_m(\chi) = 4^{m-1} \sum_{v=0}^3 \chi(-v) B_m\left(\frac{v}{4}\right) = \frac{m}{2} E_{m-1}.$$

Using $G(ix, \chi) = e^{-\pi x} (e^{\pi x/2} - e^{-\pi x/2})$ and $G(n, \chi) = \overline{\chi}(n) G(1, \chi) = 2i\chi(n)$, and simplifying give, for $\gamma\theta = \pi^2/16$, that

$$\begin{aligned} & \gamma^{-N} \sum_{n=1}^{\infty} \chi(n) \frac{\text{sech}(2n\gamma)}{n^{2N+1}} + (-\theta)^{-N} \sum_{n=1}^{\infty} \chi(n) \frac{\text{sech}(2n\theta)}{n^{2N+1}} \\ &= 2^{2N} \frac{\pi}{4} \sum_{m=0}^N (-1)^m \frac{E_{2m}}{(2m)!} \frac{E_{2N-2m}}{(2N-2m)!} \gamma^{N-m} \theta^m, \end{aligned} \quad (70)$$

where we have used the fact that $G(n, \chi) = \overline{\chi}(n) G(1, \chi)$ for primitive character χ [4, p. 168].

We remark that in the case when χ is primitive (62)–(67) have been given by Berndt [10, 12]. Formula (70) is also found in Ramanujan's Notebooks; see Entry 21 (ii) on p. 276 of Berndt [14]. Also Entries 14, 15 and 25 (vii), (viii), (ix) in [14] are special cases of (70). Moreover, since formula (63) is a generalization of Theorem 4.2 in [12] from primitive characters to Dirichlet characters, the results of Theorem 4.2 in [12] are also consequences of (63). Historical informations and literature of (70) and their results can be found in [12].

Example 15 Let $\chi_1 = \chi_2 = \chi_0$ with modulus $k = 4$ in (65). Then we have

$$G(ix, \chi_0) = e^{-\pi x} (e^{\pi x/2} + e^{-\pi x/2}), \quad G(n, \chi_0) = 2(-1)^n \cos(\pi n/2)$$

and by (69)

$$B_{2m+1}(\chi_0) = 0 \text{ and } B_{2m}(\chi_0) = 2^{2m-1} B_{2m}(1/2), \quad m \geq 0.$$

Replacing γ and θ by $\gamma/4$ and $\theta/4$, respectively, and simplifying give, for $\gamma\theta = \pi^2$, that

$$\begin{aligned} & \gamma^{-N} \sum_{n=1}^{\infty} (-1)^n \frac{\text{csch}(n\gamma)}{n^{2N+1}} - (-\theta)^{-N} \sum_{n=1}^{\infty} (-1)^n \frac{\text{csch}(n\theta)}{n^{2N+1}} \\ &= -2^{2N+1} \sum_{m=0}^{N+1} (-1)^m \frac{B_{2m}(1/2)}{(2m)!} \frac{B_{2N+2-2m}(1/2)}{(2N+2-2m)!} \gamma^{N+1-m} \theta^m. \end{aligned} \quad (71)$$

It follows for $N = 2M + 1$ that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\operatorname{csch}(n\pi)}{n^{4M+3}} = -(2\pi)^{4M+3} \sum_{m=0}^{2M+2} (-1)^m \frac{B_{2m}(1/2)}{(2m)!} \frac{B_{4M+4-2m}(1/2)}{(4M+4-2m)!},$$

which was first proved by Cauchy as cited by Berndt [13].

(71) has been also established by Berndt [13, Theorem 3.1]. Historical informations and literature of (71) and their results can be found in [13].

2. Let $A = \{f(n)\} = \{G(n, \chi_1)\} = G_1$ and $B = \{f^*(n)\} = \{G(n, \chi_2)\} = G_2$, where χ_1 and χ_2 are Dirichlet characters of modulus k . Then, $A_\alpha = \overline{\chi_1}(\alpha) G_1$ and $B_\alpha = \overline{\chi_2}(\alpha) G_2$ for $(\alpha, k) = 1$, and from (9) $\hat{A} = \{\chi_1(n)\}$ and $\hat{B} = \{\chi_2(n)\}$. So that we find from (57), (58) and (37) that

$$\begin{aligned} & k \left(A(Vz, 0; G_1, \chi_2) - \overline{\chi_1}(c) \overline{\chi_2}(b) A(z, 0; G_2, \chi_1) \right) (\chi_1(-1) + \chi_2(-1)) \\ & + \lim_{s \rightarrow 0} \left(\Gamma(s) (cz + d)^{-s} (1 + e(-s/2) \chi_2(-1)) L(s, G_2) G(0, \chi_1) \right. \\ & \quad \left. - \Gamma(s) (1 + e(-s/2) \chi_1(-1)) L(s, G_1) G(0, \chi_2) \overline{\chi_1}(-c) \right) \\ & = 2\pi i \overline{\chi_1}(c) \overline{\chi_2}(b) s(d, c; G_2, G_1) - \frac{\pi i}{c(cz + d)} B_0(G_2) B_2(G_1) - 2\pi i \overline{\chi_2}(b) \frac{cz + d}{c} B_0(G_1) P_2(0, G_2). \end{aligned} \quad (72)$$

Suppose that $\chi_1(-1) \chi_2(-1) = 1$. Thus (72) yields

$$\begin{aligned} & A(Vz, 0; G_1, \chi_2) - \overline{\chi_1}(c) \overline{\chi_2}(b) A(z, 0; G_2, \chi_1) \\ & = \frac{\pi i}{k} \overline{\chi_1}(-c) \overline{\chi_2}(b) s(d, c; G_2, G_1) + \delta(z; c, d; \chi_1, \chi_2), \end{aligned} \quad (73)$$

where

$$\delta(z; c, d; \chi_1, \chi_2) = \begin{cases} \frac{c_k(0)}{k} P_1(0, c_k) \log(cz + d), & \text{if } \chi_1 = \chi_2 = \chi_0, \\ 0, & \text{if } \chi_1 \neq \chi_0 \text{ and } \chi_2 \neq \chi_0, \end{cases} \quad (74)$$

and we have used (31) and that $G(0, \chi) = 0$ for $\chi \neq \chi_0$ and $B_0(G) = 0$.

Note that the number $P_1(0, G)$ can be evaluated as

$$\begin{aligned} P_1(0, G) &= \sum_{v=0}^{k-1} G(-v, \chi) P_1\left(\frac{v}{k}\right) = \sum_{j=1}^{k-1} \chi(j) \frac{1}{e^{-2\pi i j/k} - 1} \\ &= \frac{i}{2} \sum_{j=1}^{k-1} \chi(j) \cot(\pi j/k) - \frac{1}{2} \sum_{j=1}^{k-1} \chi(j). \end{aligned} \quad (75)$$

In fact, the numbers $P_r(0, G)$, $r \geq 1$, are closely related to the Dirichlet L -function $L(r, \chi)$. Let χ be the Dirichlet character of modulus $k \geq 2$ and put $\chi(-1) = (-1)^l$. Alkan [2, Theorem 1] shows that if l and $r \geq 1$ have the same parity, then

$$2k \frac{(-1)^{l+1} r!}{(2\pi i)^r} L(r, \chi) = \sum_{q=0}^{2[\frac{r}{2}]} \binom{r}{q} B_q S(r - q, \chi), \quad (76)$$

where $S(m, \chi) = \sum_{j=1}^k (j/k)^m G(j, \chi)$. Now, using the facts that

$$\sum_{q=0}^r \binom{r}{q} B_q x^{r-q} = B_r(x) \text{ and } B_r(1-x) = (-1)^r B_r(x)$$

the right hand side of (76) may be written

$$(-1)^r \sum_{j=0}^{k-1} G(-j, \chi) B_r\left(\frac{j}{k}\right) = (-1)^r r! k^{1-r} P_r(0, G),$$

which yields

$$P_r(0, G) = -2 \left(\frac{k}{2\pi i} \right)^r L(r, \chi), \text{ if } l \text{ and } r \geq 1 \text{ have the same parity.}$$

Furthermore since $\cot x$ is an odd function, it is seen for even Dirichlet character χ that $P_1(0, G) = 0$ if $\chi \neq \chi_0$, and $P_1(0, G) = P_1(0, c_k) = -\phi(k)/2$ if $\chi = \chi_0$. Therefore, for $Vz = -1/z$, (73) may be written explicitly as

$$\begin{aligned} A(-1/z, 0; G_1, \chi_2) + A(z, 0; G_2, \chi_1) &= -\frac{k}{\pi i} L(1, \chi_1) L(1, \chi_2), \text{ for odd } \chi_1 \text{ and } \chi_2, \\ A(-1/z, 0; G_1, \chi_2) - A(z, 0; G_2, \chi_1) &= 0, \text{ for even } \chi_1 \neq \chi_0 \text{ and } \chi_2 \neq \chi_0, \\ A(-1/z, 0; c_k, \chi_0) - A(z, 0; c_k, \chi_0) &= \frac{\phi(k)^2}{2k} \left(\frac{\pi i}{2} - \log z \right), \text{ for } \chi_1 = \chi_2 = \chi_0. \end{aligned}$$

The counterpart of (64) for $A(z, 0; G_1, \chi_2)$ is

$$A(z, 0; G_1, \chi_2) = \sum_{v=1}^{k-1} \chi_1(v) \sum_{n=1}^{\infty} \frac{\chi_2(n)}{1 - e^{2\pi i(v+nz)/k}} n^{-1}.$$

As before, let $z = \pi i/k\gamma$, where $\gamma > 0$, and determine $\theta > 0$ by $\gamma\theta = \pi^2/k^2$. Then, for $Vz = -1/z$, (73) becomes

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{v=0}^{k-1} \frac{\chi_1(v) \chi_2(n)}{1 - e^{2\pi i v/k - 2n\gamma}} n^{-1} - \chi_2(-1) \sum_{n=1}^{\infty} \sum_{v=0}^{k-1} \frac{\chi_2(v) \chi_1(n)}{1 - e^{2\pi i v/k - 2n\theta}} n^{-1} \\ &= \frac{\pi i}{k} \chi_1(-1) P_1(0, G_1) P_1(0, G_2) + \delta(z; 1, 0; \chi_1, \chi_2), \end{aligned} \quad (77)$$

where $\delta(z; 1, 0; \chi_1, \chi_2)$ is given by (74).

Example 16 Let $\chi_1 = \chi_2 = \chi$ be the Dirichlet character of modulus $k = 6$ defined by

$$\chi(n) = \begin{cases} 1, & n \equiv 1 \pmod{6}, \\ -1, & n \equiv 5 \pmod{6}, \\ 0, & (n, 6) > 1 \end{cases} \quad (78)$$

in (77). A simple calculation gives

$$\sum_{n=1}^{\infty} \frac{\chi(n) n^{-1}}{2 \cosh(2n\gamma) - 1} + \sum_{n=1}^{\infty} \frac{\chi(n) n^{-1}}{2 \cosh(2n\theta) - 1} = \frac{\pi}{2\sqrt{3}}, \text{ for } \gamma\theta = \pi^2/36$$

and

$$\sum_{n=1}^{\infty} \frac{\chi(n) n^{-1}}{2 \cosh(n\pi/3) - 1} = \frac{\pi}{4\sqrt{3}}, \text{ for } \gamma = \theta = \pi/6.$$

Example 17 Let $\chi_1 = \chi_2 = \chi$ be the primitive character of modulus $k = 3$ defined by

$$\chi(n) = \begin{cases} 0, & n \equiv 0 \pmod{3}, \\ 1, & n \equiv 1 \pmod{3}, \\ -1, & n \equiv 2 \pmod{3} \end{cases}$$

(77). Then, for $\gamma\theta = \pi^2/9$

$$\sum_{n=1}^{\infty} \frac{\chi(n) n^{-1}}{2 \cosh 2n\gamma + 1} + \sum_{n=1}^{\infty} \frac{\chi(n) n^{-1}}{2 \cosh 2n\theta + 1} = \frac{\pi}{9\sqrt{3}}.$$

Example 18 Let $\chi_1 = \chi_2 = \chi_0$ with modulus $k = 4$ in (77). Then we have, for $\gamma\theta = \pi^2$, that

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{1+e^{-(2n+1)\gamma}} - \frac{1}{1+e^{-(2n+1)\theta}} \right) = \frac{1}{8} (\log \gamma - \log \theta), \quad (79)$$

which is same with Corollary 4.3 in [13]. Differentiating both sides of (79) with respect to γ gives

$$\begin{aligned} \gamma \sum_{n=0}^{\infty} \operatorname{sech}^2 \left((2n+1) \frac{\gamma}{2} \right) + \theta \sum_{n=0}^{\infty} \operatorname{sech}^2 \left((2n+1) \frac{\theta}{2} \right) &= 1, \\ \sum_{n=0}^{\infty} \operatorname{sech}^2 \left((2n+1) \frac{\pi}{2} \right) &= \frac{1}{2\pi}, \end{aligned}$$

i.e., Corollaries 4.4 and 4.5 in [13], respectively.

3. Let $A = \{(-1)^n \chi_1(n)\}$ and $B = \{(-1)^n \chi_2(n)\}$, where χ_1 and χ_2 are Dirichlet characters of modulus k . Assume that k is even and $\chi_1(-1)\chi_2(-1) = 1$. Then, from (59),

$$\begin{aligned} \hat{A} &= \frac{\chi_1(-1)}{k} \{G(n+k/2, \chi_1)\} = \frac{\chi_1(-1)}{k} G_1^*, \\ \hat{B} &= \frac{\chi_2(-1)}{k} \{G(n+k/2, \chi_2)\} = \frac{\chi_2(-1)}{k} G_2^*. \end{aligned}$$

Since $ad - bc = 1$ and $a \equiv d \equiv 0 \pmod{k}$, b and c must be odd. It can be seen for $(k, \alpha) = 1$ that $G(\alpha n + k/2, \chi) = \overline{\chi}(\alpha) G(n+k/2, \chi)$, which implies $\hat{A}_\alpha = \overline{\chi_1}(-\alpha) G_1^*/k$ and $\hat{B}_\alpha = \overline{\chi_2}(-\alpha) G_2^*/k$. Also

$$P_m(x, A_c) = \chi_1(-c) k^{m-1} \sum_{v=0}^{k-1} (-1)^v \chi_1(v) P_m\left(\frac{v+x}{k}\right) = \chi_1(-c) P_m^*(x, \overline{\chi_1}) \quad (80)$$

and

$$\begin{aligned} s(d, c; B_b, A_c) &= \chi_1(-c) \chi_2(b) \sum_{n=1}^{ck} (-1)^n \chi_2(n) P_1\left(\frac{n}{ck}\right) P_1^*\left(\frac{dn}{c}, \overline{\chi_1}\right) \\ &= \chi_1(-c) \chi_2(b) s^*(d, c; \chi_2, \chi_1), \end{aligned} \quad (81)$$

where $P_m^*(x, \overline{\chi_1})$ and $s^*(d, c; \chi_2, \chi_1)$ are the alternating Bernoulli function and Dedekind character sum defined in [23]. Finally put

$$A_1(z, s; \chi, G^*) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^m \chi(m) G\left(n + \frac{k}{2}, \chi\right) e^{2\pi i n m z / k} n^{s-1}.$$

Now using (57) and (58) with the notations above we deduce from (37) that

$$\begin{aligned} &A_1(Vz, 0; \chi_1, G_2^*) - \chi_1(c) \chi_2(b) A_1(z, 0; \chi_2, G_1^*) \\ &= \pi i \chi_1(-c) \chi_2(b) s^*(d, c; \chi_2, \chi_1) - \frac{\pi i}{c(cz+d)} B_0^*(\overline{\chi_2}) B_2^*(\overline{\chi_1}) \\ &\quad - \frac{2\pi i}{c} (cz+d) \chi_2(-b) B_0^*(\overline{\chi_1}) P_2^*(0, \overline{\chi_2}). \end{aligned} \quad (82)$$

Notice that when $\chi_1 = \chi = \overline{\chi_2}$ is primitive, (82) has been given by Meyer [23, Theorem 3]. Similar to (64), one has

$$A_1(z, s; \chi_1, G_2^*) = \sum_{n=1}^{\infty} \frac{G\left(n + \frac{k}{2}, \chi_2\right) G\left(nz + \frac{k}{2}, \chi_1\right)}{1 - e^{2\pi i n z}} n^{s-1}.$$

Set $z = \pi i / k\gamma$, where $\gamma > 0$, and determine $\theta > 0$ by $\gamma\theta = \pi^2 / k^2$. Then, for $Vz = -1/z$, (82) becomes

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{G\left(n + \frac{k}{2}, \chi_2\right) G\left(\frac{in\gamma k}{\pi} + \frac{k}{2}, \chi_1\right)}{n(1 - e^{-2n\gamma k})} - \chi_2(-1) \sum_{n=1}^{\infty} \frac{G\left(n + \frac{k}{2}, \chi_1\right) G\left(\frac{in\theta k}{\pi} + \frac{k}{2}, \chi_2\right)}{n(1 - e^{-2n\theta k})} \\ &= \pi i P_1^*(0, \overline{\chi_2}) P_1^*(0, \overline{\chi_1}) - k\gamma B_0^*(\overline{\chi_2}) B_2^*(\overline{\chi_1}) + \theta k \chi_2(-1) B_0^*(\overline{\chi_1}) B_2^*(\overline{\chi_2}). \end{aligned} \quad (83)$$

Example 19 Let, in (83), $\chi_1 = \chi_2 = \chi$ be the primitive character of modulus $k = 4$ given by (68). From [23, Proposition 1]

$$\sum_{v=0}^{k-1} (-1)^v \chi_1(v) = 0 \text{ for even } k \text{ and } \chi \neq \chi_0$$

we have $B_0^*(\overline{\chi}) = 0$. Using $G(nz + k/2, \chi) = e^{\pi inz} (e^{\pi inz/2} - e^{-\pi inz/2})$ and $G(n + k/2, \chi) = -2i\chi(n)$, and simplifying we deduce that, for $\gamma\theta = \pi^2$

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n(e^{n\gamma/2} + e^{-n\gamma/2})} + \sum_{n=1}^{\infty} \frac{\chi(n)}{n(e^{n\theta/2} + e^{-n\theta/2})} = \frac{\pi}{8},$$

which is special case of (70).

Example 20 Let $\chi_1 = \chi_2 = \chi_0$ with modulus $k = 8$ in (83). Some arguments give $P_1^*(0, \chi_0) = 0$, $B_0^*(\chi_0) = -4$, $B_2^*(\chi_0) = -2(B_2(1/8) + B_2(3/8))$ and

$$G(nz + k/2, \chi) = -e^{2\pi inz} (e^{\pi inz/4} + e^{-\pi inz/4}) (1 + e^{\pi inz}).$$

Hence, with some simplifications, for $\gamma\theta = \pi^2/4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-12n\gamma}}{n \sinh(2n\gamma)} - \sum_{n=1}^{\infty} \frac{(-1)^n e^{-12n\theta}}{n \sinh(2n\theta)} = 32(\theta - \gamma) \left(\frac{11}{192} - \frac{13}{192} \right) = \frac{1}{3}(\gamma - \theta).$$

Example 21 Let, in (83), $\chi_1 = \chi_2 = \chi$ be the Dirichlet character of modulus $k = 6$ given by (78). Using $B_0^*(\overline{\chi}) = 0$ and $G(nz + k/2, \chi) = e^{\pi inz} (e^{2\pi inz/3} - e^{-2\pi inz/3})$, and simplifying we deduce, for $\gamma\theta = \pi^2/9$, that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(2\pi n/3) \cosh(n\gamma)}{n(2 \cosh(2n\gamma) + 1)} + \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2\pi n/3) \cosh(n\theta)}{n(2 \cosh(2n\theta) + 1)} = \frac{\pi}{9}.$$

7 Some examples

In this final section, we deal with Theorem 8 for the special values of $A = \{f(n)\}$ and $B = \{f^*(n)\}$. χ_1 and χ_2 will denote Dirichlet characters of modulus k , and χ_0 still stands for principle character.

1. When $A = B = I$,

$$s(d, c; I, I) = \sum_{n=1}^c P_1\left(\frac{n}{c}\right) P_1\left(\frac{dn}{c}\right) = \sum_{n=1}^{c-1} \left(\left(\frac{n}{c}\right)\right) \left(\left(\frac{dn}{c}\right)\right) + \frac{1}{4} = s(d, c) + \frac{1}{4}$$

and from (43) we have $s(-c, d; I, I) = -s(c, d) + 1/4$. So, Theorem 8 reduces to the reciprocity formula for the classical Dedekind sums given by (1).

2. Let $A = \chi_1 = \{\chi_1(n)\}$ and $B = \chi_2 = \{\chi_2(n)\}$. Then, using that $s(c, d; A_c, B_b) = \chi_1(c) \chi_2(b) s(c, d; \chi_1, \chi_2)$ and $P_m(0, A_c) = \chi(c) P_m(0, \chi_1) = \chi(c) B_m(\chi_1)/m!$, we have

$$\begin{aligned} & s(c, d; \chi_1, \chi_2) + s(d, c; \chi_2, \chi_1) \\ &= -B_1(\chi_1)B_1(\chi_2) + \left(\frac{\overline{\chi_1}(c)}{c} + c\right) \frac{\overline{\chi_2}(b)}{2d} B_0(\chi_2) B_2(\chi_1) + \frac{d}{2c} \overline{\chi_1}(c) B_0(\chi_1) B_2(\chi_2), \end{aligned} \quad (84)$$

where we assume that $\chi_1(-1)\chi_2(-1) = 1$, otherwise $s(c, d; \chi_1, \chi_2) = 0$. This can be simplified by using (61).

In the case when χ_1 and χ_2 are primitive, $s(d, c; \chi_1, \chi_2)$ arises in the transformation formula derived by Berndt [10] and is generalized in [18]. The sum $s(d, c; \chi, \overline{\chi})$ is first defined in [6] and [11, Section 7] according to χ is primitive or non-principle Dirichlet character.

3. Let $A = \chi_1 = \{\chi_1(n)\}$ and $B = G_2 = \{G(n, \chi_2)\}$. Using that $B_0(B) = B_0(G_2) = 0$, $P_m(x, B_b) = \overline{\chi_2}(b) P_m(x, G_2)$ and $s(d, c; B_b, A_c) = \chi_1(c) \overline{\chi_2}(b) s(d, c; G_2, \chi_1)$, Theorem 8 becomes

$$s(c, d; \chi_1, G_2) + s(d, c; G_2, \chi_1) = -\chi_1(-1) \chi_2(-1) P_1(0, \chi_1) P_1(0, G_2)$$

when χ_1 and χ_2 are non-principle, and

$$s(c, d; \chi_0, c_k) + s(d, c; c_k, \chi_0) = \frac{d}{c} B_0(\chi_0) P_2(0, c_k)$$

when $\chi_1 = \chi_2 = \chi_0$.

Note that for $k = 2$, provided that d is even with $(d, c) = 1$, the sums $s(c, d; \chi_0, c_2)$ and $s(d, c; c_2, \chi_0)$ can be expressed in terms of the sums $s(c, d)$ and $s_2(d, c)$ as

$$\begin{aligned} s(c, d; \chi_0, c_2) &= 2s(c, 2d) - 3s(c, d) + s(2c, d), \\ s(d, c; c_2, \chi_0) &= s_2(2d, 2c) - s_2(d, 2c), \end{aligned} \tag{85}$$

where $s_2(d, c)$ is one of the Hardy–Berndt sums defined by [13, 19]

$$s_2(d, c) = \sum_{n=1}^{c-1} (-1)^n P_1\left(\frac{n}{c}\right) P_1\left(\frac{dn}{c}\right).$$

These expressions follow from the fact $P_1(x + 1/2) = P_1(2x) - P_1(x)$.

Remark 22 If we set $A = G_2 = \{G(n, \chi_2)\}$ and $B = \chi_1 = \{\chi_1(n)\}$, then with the use of $B_0(A) = B_0(G_2) = 0$, $P_m(x, A_c) = \overline{\chi_2}(c) P_m(x, G_2)$ and $s(d, c; B_b, A_c) = \chi_1(b) \overline{\chi_2}(c) s(d, c; \chi_1, G_2)$, Theorem 8 would become

$$s(c, d; G_2, \chi_1) + s(d, c; \chi_1, G_2) = -\chi_1(-1) \chi_2(-1) P_1(0, \chi_1) P_1(0, G_2)$$

when χ_1 and χ_2 are non-principle, and

$$s(c, d; c_k, \chi_0) + s(d, c; \chi_0, c_k) = \left(\frac{1}{c} + c\right) \frac{1}{d} B_0(\chi_0) P_2(0, c_k)$$

when $\chi_1 = \chi_2 = \chi_0$.

4. Let $A = \{G(n, \chi_1)\} = G_1$ and $B = \{G(n, \chi_2)\} = G_2$. One has $A_c = \{G(cn, \chi_1)\} = \{\overline{\chi_1}(c) G(n, \chi_1)\} = \overline{\chi_1}(c) G_1$ and $B_c = \overline{\chi_2}(c) G_2$ for $(c, k) = 1$. Thus, we have $B_0(A) = B_0(B) = 0$, $P_1(0, A_{-c}) = \overline{\chi_1}(-c) P_1(0, G_1)$ and $s(d, c; B_b, A_c) = \overline{\chi_1}(c) \overline{\chi_2}(b) s(d, c; G_2, G_1)$. Then, for $\chi_1 = \chi_2 = \chi_0$,

$$s(c, d; c_k, c_k) + s(d, c; c_k, c_k) = \frac{1}{4} \phi(k) \phi(k)$$

and for $\chi_1 \neq \chi_0$ and $\chi_2 \neq \chi_0$,

$$s(c, d; G_1, G_2) + s(d, c; G_2, G_1) = -P_1(0, G_1) P_1(0, G_2).$$

In particular, for $k = 2$, provided that d is even with $(d, c) = 1$,

$$s(d, c; c_2, c_2) = 2s_2(d, 2c) - s_2(2d, 2c) + \frac{1}{4}.$$

5. $A = \{(-1)^n \chi_1(n)\}$ and $B = \{(-1)^n \chi_2(n)\}$. If k is even, then the sequences A and B have period $h = k$. If k is odd, then A and B have period $h = 2k$. In both cases b and c must be odd since $ad - bc = 1$ and $a \equiv d \equiv 0 \pmod{h}$. Using (80) and (81) in Theorem 8 gives

$$\begin{aligned} & s^*(c, d; \chi_1, \chi_2) + \chi_1(-1) \chi_2(-1) s^*(d, c; \chi_2, \chi_1) \\ &= -\chi_2(-1) P_1^*(0, \overline{\chi_2}) P_1^*(0, \overline{\chi_1}) + \frac{1}{2dc} \overline{\chi_1}(-c) \overline{\chi_2}(-b) B_0^*(\overline{\chi_2}) B_2^*(\overline{\chi_1}) \\ &+ \frac{d}{c} \overline{\chi_1}(c) B_0^*(\overline{\chi_1}) P_2^*(0, \overline{\chi_2}) + \frac{c}{d} \chi_1(-1) \overline{\chi_2}(-b) B_0^*(\overline{\chi_2}) P_2^*(0, \overline{\chi_1}). \end{aligned}$$

For non-principle characters χ_1 and χ_2 , this reduces to

$$s^*(c, d; \chi_1, \chi_2) + \chi_1(-1) \chi_2(-1) s^*(d, c; \chi_2, \chi_1) = -\chi_2(-1) P_1^*(0, \overline{\chi_2}) P_1^*(0, \overline{\chi_1}) \quad (86)$$

since $B_0^*(\overline{\chi_1}) = B_0^*(\overline{\chi_2}) = 0$. Remark that (86) has been obtained by Meyer [23, Theorem 4] when $\overline{\chi_1} = \chi_2 = \chi$ is primitive character of modulus $h = k$ (k even).

6. Setting $A = \{e^{2\pi i n/k}\}$ and $B = \hat{A} = \{\hat{f}(n)\}$. In view of (9), we have

$$\hat{f}(n) = \frac{1}{k} \sum_{j=0}^{k-1} e^{2\pi i (1-n)j/k} = \begin{cases} 1, & \text{if } n \equiv 1 \pmod{k} \\ 0, & \text{if } n \not\equiv 1 \pmod{k}. \end{cases}$$

Then using that $B_0(A) = 1$, $B_0(\hat{A}) = 0$, $P_1(0, \hat{A}_{-b}) = -P_1(c/k)$, $P_2(0, \hat{A}_b) = P_2(c/k)$, $P_1(0, A_{-c}) = -1/2 - (i/2) \cot(\pi c/k)$ and

$$P_1\left(\frac{cn}{d}, \hat{A}_b\right) = \sum_{v=0}^{k-1} \hat{f}(-bv) P_1\left(\frac{v + cn/d}{k}\right) = P_1\left(\frac{c + cn/d}{k}\right),$$

reciprocity formula becomes

$$s\left(c, d; A_c, \hat{A}_b\right) + s\left(d, c; \hat{A}_b, A_c\right) = -P_1(c/k) \left(\frac{1}{2} + \frac{i}{2} \cot \frac{\pi c}{k}\right) + \frac{d}{c} P_2\left(\frac{c}{k}\right).$$

In this case the periodic Dedekind sums take form

$$\begin{aligned} s\left(c, d; A_c, \hat{A}_b\right) &= \sum_{n=1}^{dk} e^{2\pi i cn/k} P_1\left(\frac{n}{dk}\right) P_1\left(\frac{c(d+n)}{dk}\right), \\ s\left(d, c; \hat{A}_b, A_c\right) &= \sum_{\mu=0}^{c-1} P_1\left(\frac{\mu}{c} - \frac{1}{k}\right) P_1\left(\frac{dk\mu}{c}, A_c\right) \end{aligned}$$

since $d \equiv 0 \pmod{k}$.

If we take $k = 2$, provided that c is odd, then we have

$$s\left(c, d; A_c; \hat{A}_b\right) + s\left(d, c; \hat{A}_b; A_c\right) = -\frac{d}{24c}, \quad (87)$$

where we have used that $P_1(c/2) = P_1(1/2) = 0$ and $2P_2(c/2) = 2P_2(1/2) = B_2(1/2) = (1/2)^2 - 1/2 + 1/6 = -1/12$.

Remark 23 Using $P_1(x + \frac{1}{2}) = P_1(2x) - P_1(x)$, we have

$$s\left(c, d; A_c; \hat{A}_b\right) = s_2(2c, 2d) - s_2(c, 2d)$$

and from [15, Eq.(3.4)]

$$\begin{aligned} s\left(d, c; \hat{A}_b; A_c\right) &= 2 \sum_{\mu=1}^{c-1} P_1\left(\frac{\mu}{c} + \frac{1}{2}\right) P_1\left(\frac{d\mu}{c}\right) - \sum_{\mu=1}^{c-1} P_1\left(\frac{\mu}{c} + \frac{1}{2}\right) P_1\left(\frac{2d\mu}{c}\right) \\ &= 2s(d, c) + s_3\left(\frac{d}{2}, c\right) - s(2d, c) - \frac{1}{2}s_3(d, c), \end{aligned}$$

where $s_3(d, c)$ is one of the Hardy–Berndt sums defined by [13, 19]

$$s_3(d, c) = \sum_{n=1}^{c-1} (-1)^n P_1\left(\frac{dn}{c}\right).$$

Then (87) becomes

$$2s(d, c) - s(2d, c) + s_2(2c, 2d) - s_2(c, 2d) + s_3(d/2, c) - \frac{1}{2}s_3(d, c) = -\frac{d}{24c}.$$

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